MATH 258 Differential Equations for Mechanical Engineering

Learning Goals, Lecture Schedule & Exercises

MATH 258 (in MECH 221) is an introduction to ordinary differential equations with applications in dynamics and electric circuits. The course is divided into five main topics: first order equations, second order equations, Laplace transforms, linear systems, and non-linear systems of equations.

Learning Goals

First Order Equations

- Solve separable equations $y' = f(t)g(y)$ by separation of variables
- Solve linear equations $y' + p(t)y = q(t)$ with the integrating factor
- Derive first order equations to model growth, decay, fluid mixing, friction and drag
- Sketch slope fields and describe qualitative behaviour of solutions
- Sketch phase portraits of autonomous equations $y' = f(y)$ and classify equilibrium solutions
- Use MATLAB to implement Euler's method and numerically approximate solutions
- Use MATLAB to plot slope fields and solutions of equations

Second Order Equations

- Solve second order homogeneous equations with constant coefficients $ay'' + by' + cy = 0$
  - Identify the characteristic polynomial $p(s) = as^2 + bs + c$
  - Compute the solution when $p(s)$ has real distinct roots, complex roots, or real repeated roots
- Solve second order non-homogeneous equations with constant coefficients $ay'' + by' + cy = f(t)$
  - Find the complimentary solution $y_c(t)$ of the homogeneous equation $ay'' + by' + cy = 0$
  - Find any particular solution $y_p(t)$ of the equation $ay'' + by' + cy = f(t)$ using the method of undetermined coefficients (aka guess and check)
  - Combine $y_c(t)$ and $y_p(t)$ to form the general solution $y(t) = y_p(t) + y_c(t)$
- Derive and solve equations for mass-spring-damper systems and RLC circuits
- Classify second order vibration equations: forced, unforced, undamped, over-damped, under-damped, critically damped
- Derive general formulas for forced oscillation and describe resonance and beats

Laplace Transforms

- Compute the Laplace transform from the definition

$$\mathscr{L}\{f(t)\} = F(s) = \int_0^\infty e^{-st} f(t)dt$$

  when $f(t)$ is a combination of elementary functions: 1, t, t^n, $e^{-\lambda t}$, $u(t-a)$, $\sin(\omega t)$, $\cos(\omega t)$

- Compute $\mathscr{L}\{f(t)\} = F(s)$ and the inverse $\mathscr{L}^{-1}\{F(s)\} = f(t)$ using the Laplace transfrom tables

- Transform differential equations in the (time) t-domain into algebraic equations in the (frequency) s-domain using the derivative properties:

$$\mathscr{L}\{y'(t)\} = sY(s) - y(0)$$
$$\mathscr{L}\{y''(t)\} = s^2Y(s) - sy(0) - y'(0)$$

- Apply the Laplace transform to circuit analysis

- Find the steady state frequency response: if $Y(s) = H(s)F(s)$ and $f(t) = A\sin(\omega t)$ then

$$\lim_{t\to\infty} = A|H(j\omega)|\sin(\omega t + \phi) \ , \quad \phi = \arg(H(j\omega))$$

Linear Systems of Equations

- Write any system of higher order ODEs as a first order system of ODEs

- Solve a linear homogeneous system of equations with constant coefficient matrix $\dot{\boldsymbol{x}} = A\boldsymbol{x}$ by the eigenvalue method for all cases: distinct real eigenvalues, complex eigenvalues, repeated eigenvalues

- Sketch phase planes of 2D linear systems: eigenvectors, nullclines, trajectories and flow directions

- Classify the origin of a linear system: sink, source, saddlepoint, spiral sink, spiral source or centre

Non-Linear Systems Equations

- Sketch phase planes for non-linear scalar equations and classify equilibrium solutions as stable, unstable or semistable

- Given a 2D system of autonomous non-linear equations:

$$\dot{x}_1 = f_1(x_1, x_2)$$
$$\dot{x}_2 = f_2(x_1, x_2)$$

  - Sketch the nullclines, indicate the flow directions and identify the steady states
  - Compute the Jacobian $J_{\boldsymbol{x}^*}$ at a steady state $\boldsymbol{x}^* = (x_1^*, x_2^*)$

$$J_{\boldsymbol{x}^*} = \left.\begin{pmatrix} \dfrac{\partial f_1}{\partial x_1} & \dfrac{\partial f_1}{\partial x_2} \\ \dfrac{\partial f_2}{\partial x_1} & \dfrac{\partial f_2}{\partial x_2} \end{pmatrix}\right|_{x_1=x_1^*, x_2=x_2^*}$$

  - Classify steady states $\boldsymbol{x}^*$ according to the eigenvalues of the Jacobian $J_{\boldsymbol{x}^*}$ in the case $\boldsymbol{x}^*$ is hyperbolic: the real parts of both eigenvalues of $J_{\boldsymbol{x}^*}$ are nonzero
  - Sketch trajectories in the phase plane following nullclines and classification of steady states

Lecture Schedule & Exercises

Exercises are taken from our course textbooks: Notes on Diffy Qs by Jiri Lebl [JL] and Elementary Differential Equations by William F. Trench [WFT].

Lectures	Topics	Exercises
1	Introduction to ODEs, solving first order separable equations	JL 1.3: 1-10,101-105 WFT 2.2: 1-29 (odd), 30
2	Linear equations and integrating factors	JL 1.4: 4-8 WFT 2.1: 1-37 (odd)
3	Applications of first order equations	JL 1.4: 9-12, 103-105 WFT 4.2: 1, 4, 7, 9, 12, 15, 18, 21 WFT 4.3: 1, 4, 7, 10, 13, 16
4	Graphical methods: slope fields, phase portraits of autonomous equations	JL 1.2: 1-6, 101-104 JL 1.6: 3-5, 101, 102 WFT 1.3: 1, 3, 5, 7, 13, 15, 17, 19
5	Numerical methods: Euler's method applied to first order equations	JL 1.7: 3-6 WFT 3.1: 2, 5, 6
6 + 7	Second order homogeneous equations with constant coefficients	JL 2.2: 6-14, 101-105 WFT 5.1: 1, 2, 3, 38, 39, 40 WFT 5.2: 1-27 (odd)
8 + 9	Second order non-homogeneous equations with constant coefficients	JL 2.5: 1-3, 9, 102 WFT 5.3: 1-7, 16-23, 33-39 WFT 5.4: 1, 5, 9, 13, 17, 21, 25, 29 WFT 5.5: 1, 3, 7, 11, 15, 17, 22, 24, 26, 28, 30, 32, 40
10	Mass-spring-damper systems, rotational vibrations and RLC circuits	JL 2.4: 4-6, 101, 102 WFT 6.1: 1-9, 14 WFT 6.2: 1-12, 22, 23, 24 WFT 6.3: 1-17 (odd)
11	Forced oscillation, resonance and beats	See notes on Canvas
12	Introduction to Laplace transforms, definition and examples	WFT 8.1: 1, 2
13 + 14	Properties of Laplace transforms	JL 6.1: 5-15, 101-104 WFT 8.2: 1-8 WFT 8.4: 1, 5, 9, 13, 17, 19, 23, 27
15	Application of Laplace transforms to ODEs	JL 6.2: 3-6, 8, 102 WFT 8.3: 1-37 (odd) WFT 8.5: 1, 5, 9, 13, 17

16 + 17	Applications: dynamics and impact response, circuit analysis and frequency response	See notes on Canvas
18	Mid-semester review	
19	Introduction to first order systems and eigenvalue method	JL 3.1: 4, 5, 103, 104 WFT 10.1: 1, 2, 3, 5 WFT 10.2: 1-5 (part (a) only)
20	Eigenvalue method for systems of equations	JL 3.4: 5-11, 101-104 WFT 10.4: 1-27 (odd), 29-34 WFT 10.5: 1-4, 13-16 WFT 10.6: 1-6, 17-22, 29-34
21 + 22	Phase portraits of 2D systems	JL 3.5: 1, 101, 102
23	Application of linear systems to circuit analysis	See notes on Canvas
24	Introduction to non-linear systems of equations	See notes on Canvas
25	Jacobians of 2D systems and classifying steady states	JL 8.2: 1, 3-5, 101
26 + 27	Phase portraits of 2D non-linear systems	JL 8.1: 1-6, 101, 102
28	Final review	