MATH 307 Problem Book

UBC Math

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Contents

Linear Systems

Solving Linear Systems

1. **True** or **False**: If A is an $m \times n$ matrix such that $\text{rank}(A) = m$, then $A\boldsymbol{x} = \boldsymbol{b}$ has a unique solution for any $\boldsymbol{b} \in \mathbb{R}^m$.

[Answer]

2. Determine all values k such that $A\boldsymbol{x} = \boldsymbol{b}$ has a unique solution where

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 4 & k \end{bmatrix} \qquad \boldsymbol{b} = \begin{bmatrix} 3 \\ 4 \\ 6 \end{bmatrix}$$

[Answer]

3. **True** or **False**: If $A\boldsymbol{x} = \boldsymbol{b}$ has a unique solution, then A must be a square matrix.

[Answer]

4. **True** or **False**: An underdetermined system of linear equations $A\boldsymbol{x} = \boldsymbol{b}$ where A is an $m \times n$ matrix with $m < n$ always has at least one solution.

[Answer]

LU Decomposition

1. Consider the matrix

$$A = \begin{bmatrix} -3 & 1 & 2 & 0 \\ 3 & 1 & -2 & 1 \\ -6 & 2 & 5 & 1 \\ -9 & 3 & 4 & 2 \end{bmatrix}$$

   (a) Find the LU decomposition of A.

   (b) Compute $\det(A)$.

[Answer]

2. **True** or **False**: If A is a $n \times n$ lower triangular matrix with LU decomposition $A = LU$, then U is a diagonal matrix.

[Answer]

3. Consider the matrix

$$A = \begin{bmatrix} 2 & 1 & 1 & 0 \\ 8 & 3 & 8 & 2 \\ -4 & -3 & 5 & -1 \\ 2 & -2 & 7 & 11 \end{bmatrix}$$

   (a) Find the LU decomposition of A.

(b) Compute $\det(A)$.

[Answer]

4. Consider the matrix

$$A = \begin{bmatrix} 1 & 1 & -2 & 1 \\ -4 & -6 & 11 & -6 \\ -3 & -5 & 11 & -7 \\ 2 & 4 & -15 & 15 \end{bmatrix}$$

(a) Find the LU decomposition of A.

(b) Compute $\det(A)$.

[Answer]

5. **True** or **False**: Let A be an $m \times n$ matrix with LU decomposition $A = LU$. If U has one or more rows of zeros, then $A\boldsymbol{x} = \boldsymbol{b}$ has infinitely many solutions for any $\boldsymbol{b} \in \mathbb{R}^m$.

[Answer]

6. **True** or **False**: If A is a lower triangular matrix with LU decomposition $A = LU$, then $L = A$ and $U = I$ (where I is the identity matrix).

[Answer]

7. **True** or **False**: If A is a unit lower triangular matrix with LU decomposition $A = LU$, then $L = A$ and $U = I$ (where I is the identity matrix).

[Answer]

8. Consider the matrix

$$A = \begin{bmatrix} -4 & 3 & -3 & 1 \\ -4 & 7 & 2 & 0 \\ 16 & -24 & 2 & -2 \\ 12 & 3 & 19 & -9 \end{bmatrix}$$

(a) Find the LU decomposition of A.

(b) Compute $\det(A)$.

[Answer]

9. **True** or **False**: Suppose A_1 and A_2 are $m \times n$ matrices with LU decompositions $A_1 = L_1U$ and $A_2 = L_2U$ where U is the same matrix in both decompositions but $L_1 \neq L_2$. Then $\text{rank}(A_1) = \text{rank}(A_2)$.

[Answer]

10. Consider the matrix

$$A = \begin{bmatrix} -3 & 2 & 1 & 0 \\ -6 & 6 & 3 & -1 \\ 0 & 4 & 4 & -6 \\ 3 & -2 & 0 & -5 \end{bmatrix}$$

(a) Compute the LU decomposition of A.

(b) Compute $\det(A)$.

[Answer]

11. Solve $A\boldsymbol{x} = \boldsymbol{b}$ where

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 4 & 1 \end{bmatrix} \begin{bmatrix} -2 & 3 & 0 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & -1 & -1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad \boldsymbol{b} = \begin{bmatrix} 5 \\ -4 \\ 1 \\ -2 \end{bmatrix}$$

[Answer]

12. Compute the LU decomposition of the matrix

$$A = \begin{bmatrix} 1 & -2 & 2 & 1 \\ 2 & -6 & 6 & 3 \\ -3 & 5 & -5 & -3 \end{bmatrix}$$

[Answer]

13. **True** or **False**: A_1 and A_2 are $m \times n$ matrices such that $A_1 = LU_1$ and $A_2 = LU_2$. In other words, the unit lower triangular matrix L is the same in both LU decompositions (however the upper triangular matrices U_1 and U_2 are different). Then $L(U_1 + U_2)$ is the LU decomposition of $A_1 + A_2$.

[Answer]

14. **True** or **False**: Let $n \geq 3$ and consider the $n \times n$ matrix A with ones on the main diagonal, ones on the upper and lower diagonals and zeros everywhere else:

$$A = \begin{bmatrix} 1 & 1 & & & & \\ 1 & 1 & 1 & & & \\ & 1 & 1 & 1 & & \\ & & \ddots & \ddots & \ddots & \\ & & & 1 & 1 & 1 \\ & & & & 1 & 1 \end{bmatrix}$$

The LU factorization of A does not exist.

[Answer]

15. Let A be an $m \times n$ matrix such that the LU decomposition exists. What is the maximum number of elementary row operations required to compute the LU decomposition of A?

[Answer]

16. **True** or **False**: Let $A = LU$ be the LU decomposition of a (nonzero) matrix A. Then L is an invertible matrix.

[Answer]

17. Let

$$A = \begin{bmatrix} 1 & a & b \\ c & d & e \end{bmatrix}$$

where $a, b, c, d, e \in \mathbb{R}$.

(a) Determine the matrices L and U in the LU decomposition of A.

(b) Under what conditions on a, b, c, d, e does A have rank 1?

[Answer]

18. Consider the matrix

$$A = \begin{bmatrix} 2 & -1 & 1 \\ 4 & 1 & 5 \\ 2 & 1 & 3 \end{bmatrix}$$

(a) Compute the LU decomposition of A.

(b) Compute $\det(A)$.

[Answer]

19. Let $A = LU$ and $\boldsymbol{b} \in \mathbb{R}^3$ where

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -1 & 1 & 1 \end{bmatrix} \qquad U = \begin{bmatrix} 2 & 3 & 1 \\ 0 & 1 & 5 \\ 0 & 0 & -3 \end{bmatrix} \qquad \boldsymbol{b} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

(a) Compute $\det(A)$.

(b) Solve $A\boldsymbol{x} = \boldsymbol{b}$.

(c) Find the top left entry of the matrix A^{-1}.

[Answer]

Matrix Norm and Condition Number

1. **True** or **False**: If A is an invertible $n \times n$ matrix such that $\|A\boldsymbol{x}\| \leq \|\boldsymbol{x}\|$ for all $\boldsymbol{x} \in \mathbb{R}^n$, then $\|A^{-1}\| \geq 1$.

   [Answer]

2. Determine $\|A\|$ for the matrix

   $$A = \frac{1}{2}\begin{bmatrix} \sqrt{3} & -1 \\ 1 & \sqrt{3} \end{bmatrix}\begin{bmatrix} 5 & 0 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}$$

   Hint: rotation in $\mathbb{R}^2$ by angle θ corresponds to matrix multiplication by

   $$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

   [Answer]

3. **True** or **False**: If A is any invertible matrix, then $\text{cond}(A) \geq 1$.

   [Answer]

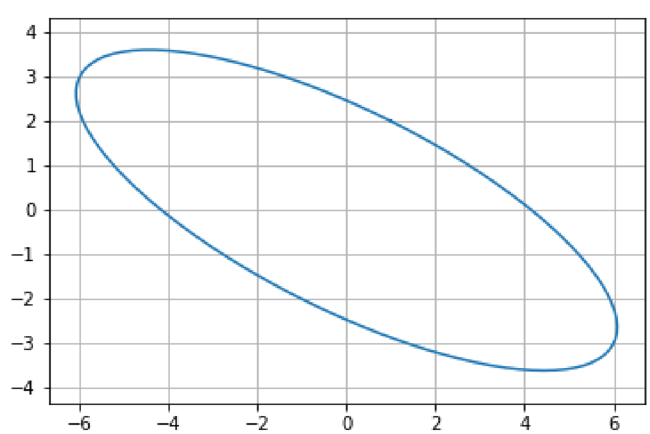
4. Determine (approximately) the condition number of the matrix

   $$A = \begin{bmatrix} c & 1 & & & \\ 1 & c & 1 & & \\ & \ddots & \ddots & \ddots & \\ & & 1 & c & 1 \\ & & & 1 & c \end{bmatrix}$$

   where c is very large positive number.

   [Answer]

5. The figure below shows the image of the unit circle in $\mathbb{R}^2$ under the linear transformation A. Determine $\text{cond}(A)$.



[Answer]

6. Let A be a $n \times n$ matrix, and let $\boldsymbol{x}_1, \boldsymbol{x}_2 \in \mathbb{R}^n$ such that $\|A\boldsymbol{x}_1\| = 5$, $\|A\boldsymbol{x}_2\| = 15$, $\|\boldsymbol{x}_1\| = 2$ and $\|\boldsymbol{x}_2\| = 1$. Find a value $C > 1$ such that $C \leq \text{cond}(A)$.

[Answer]

7. Let a and b be nonzero numbers and consider the matrix

$$A = \begin{bmatrix} a & b \\ -b & a \end{bmatrix}$$

   (a) Compute $\|A\|$.

   (b) Compute $\text{cond}(A)$.

[Answer]

8. **True** or **False**: If A is an invertible matrix then $\|A^{-1}\| \leq \|A\|$.

[Answer]

9. **True** or **False**: If A is an invertible matrix then $\|A\|\|A^{-1}\| \geq 1$.

[Answer]

10. **True** or **False**: Let A be a $m \times n$ matrix. If $\|A\boldsymbol{x}\| \leq \|\boldsymbol{x}\|$ for all $\boldsymbol{x} \in \mathbb{R}^n$ then $\text{cond}(A) = 1$.

[Answer]

11. **True** or **False**: If A is any invertible matrix then $\text{cond}(A) = \text{cond}(A^{-1})$.

[Answer]

12. Find a 2×2 matrix A satisfying $||A|| = ||A^{-1}|| = 1$ such that A is not the identity matrix or show that such a matrix does not exist.

[Answer]

13. Consider a matrix of the form

$$A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

with $0 \leq \theta < 2\pi$. The matrix A acts on a vector in $\mathbb{R}^2$ by rotating it an angle θ counterclockwise around the origin. Compute $\|A\|$ and $\text{cond}(A)$.

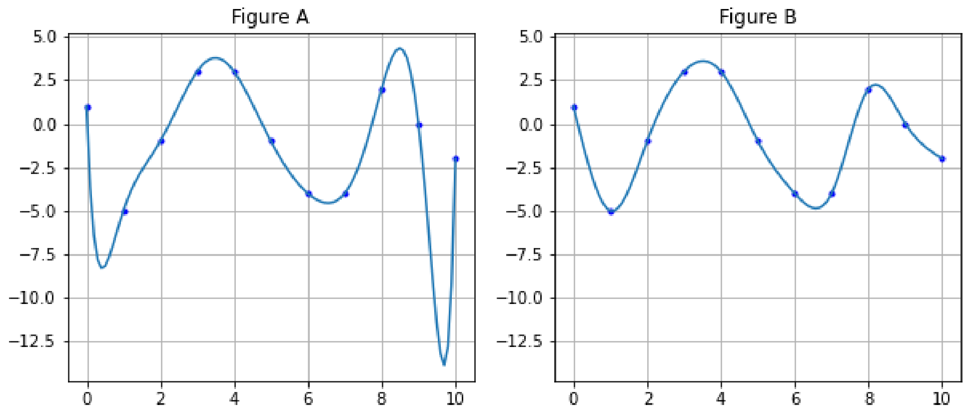
[Answer]

14. Find a matrix A such that $\text{cond}(A) = 2\|A\|$.

[Answer]

Interpolation

1. The figures show different interpolating functions for the same dataset:



Determine which figure corresponds to polynomial interpolation and which corresponds to cubic spline interpolation. Justify your answer.

[Answer]

2. Let $p(t)$ be the natural cubic spline which interpolates the data

$$(0,1)\ ,\ (1,3)\ ,\ (2,8)\ ,\ (3,10)\ ,\ (4,9)\ ,\ (5,-1)\ ,\ (6,-17)$$

Suppose the coefficient matrix of $p(t)$ is

$$\begin{bmatrix} 1 & -2 & 1 & a_4 & 1 & 1 \\ 0 & 3 & -3 & b_4 & -6 & -3 \\ 1 & 4 & 4 & c_4 & -5 & -14 \\ 1 & 3 & 8 & 10 & 9 & -1 \end{bmatrix}$$

(a) Determine the coefficients a_4, b_4, c_4.

(b) Determine the value $p''(2.5)$.

[Answer]

3. **True** or **False**: There exists a unique polynomial $p(t)$ of degree 2 (or less) such that

$$p(-1) = p'(0) = p(1) = 0$$

[Answer]

4. Consider 11 data points $(t_0, y_0), \dots, (t_{10}, y_{10})$ such that $t_k - t_{k-1} = 1$ for each $k = 1, \dots, 10$. Suppose the coefficient matrix of the corresponding natural cubic spline is given by

$$\begin{bmatrix} -1 & 2 & 6 & -9 & -1 & 3 & 2 & 8 & 3 & -13 \\ 0 & -3 & 3 & 21 & -6 & -9 & 0 & 6 & 30 & 39 \\ 2 & -1 & -1 & 23 & 38 & 23 & \square & 20 & 56 & 125 \\ -7 & -6 & -8 & 0 & 35 & 66 & 83 & 99 & 133 & 222 \end{bmatrix}$$

Determine the missing value $\square$.

[Answer]

5. Setup (but do **not** solve) a linear system $A\boldsymbol{x} = \boldsymbol{b}$ such that the solution

$$\boldsymbol{x} = \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}$$

determines the unique function of the form

$$f(t) = a\sin(\pi t) + b\cos(\pi t) + c\sin(2\pi t) + d\cos(2\pi t)$$

which interpolates the data $(0, y_0)$, $(1/4, y_1)$, $(1/2, y_2)$, $(3/4, y_3)$. The system depends on y_0, y_1, y_2, y_3.

[Answer]

6. Find all polynomials $p(t)$ of degree 3 (or less) such that

$$p(1) = p(-1) \qquad p(-2) = -7p(0) \qquad p'(1) = 3p'(-1) \qquad 5p''(1) = -7p''(-1)$$

[Answer]

7. **True** or **False**: The matrix

$$\begin{bmatrix} 5 & -1 & -1 & 2 & 1 \\ 1 & 3 & 3 & -1 & 2 \\ -4 & 5 & 4 & 0 & 0 \\ 3 & 2 & 6 & 8 & -7 \end{bmatrix}$$

is the coefficient matrix of a natural cubic spline for 6 data points $(t_0, y_0), \dots, (t_5, y_5)$ such that $t_k - t_{k-1} = 1$ for each $k = 1, \dots, 5$.

[Answer]

8. Consider $N+1$ points $(t_0, y_0), \dots, (t_N, y_N)$. Suppose we want to construct an interpolating function $p(t)$ defined piecewise by N functions $p_1(t), \dots, p_N(t)$ such that each $p_k(t)$ is defined on the interval $[t_{k-1}, t_k]$. Determine the number of equations the functions $p_1(t), \dots, p_N(t)$ must satisfy to guarantee that $p(t)$ interpolates the data and $p'(t)$, $p''(t)$ and $p'''(t)$ are continuous.

[Answer]

9. Find the unique function of the form

$$p(t) = a + bt^2 + ct^3$$

such that $p(-1) = 1$, $p'(1) = 0$ and $p''(2) = -1$.

[Answer]

10. Suppose $p(t)$ is a natural cubic spline with coefficient matrix

$$C = \begin{bmatrix} 2 & -3 & -2 & 2 & 0 & 1 \\ 0 & 6 & -3 & -9 & -3 & -3 \\ -1 & 5 & c_3 & c_4 & -16 & -22 \\ 1 & 2 & d_3 & d_4 & 2 & -17 \end{bmatrix}$$

such that $p(t)$ interpolates $(t_0, y_0), \dots, (t_6, y_6)$ where

$$t_0 = 0 \ , \ t_1 = 1 \ , \ t_2 = 2 \ , \ t_3 = 3 \ , \ t_4 = 4 \ , \ t_5 = 5 \ , \ t_6 = 6$$

and

$$p_k(t) = a_k(t - t_{k-1})^3 + b_k(t - t_{k-1})^2 + c_k(t - t_{k-1}) + d_k$$

Determine the coefficients c_3, d_3, c_4, d_4.

[Answer]

11. Let $a, b \in \mathbb{R}$ such that $0 < a < b$. Find $\boldsymbol{c}$ such that $A\boldsymbol{c} = \boldsymbol{y}$ where

$$A = \begin{bmatrix} 1 & -b & b^2 & -b^3 & b^4 \\ 1 & -a & a^2 & -a^3 & a^4 \\ 1 & 0 & 0 & 0 & 0 \\ 1 & a & a^2 & a^3 & a^4 \\ 1 & b & b^2 & b^3 & b^4 \end{bmatrix} \qquad \boldsymbol{y} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

Hints: (1) A is a Vandermonde matrix; (2) it is possible to determine $\boldsymbol{c}$ without using Gaussian elimination to solve the system.

[Answer]

12. Find the unique polynomial $p(t)$ of degree (at most) 3 that interpolates the data points $(-1, 0), (0, 1), (1, 2)$ and also satisfies $p'(1) = 5$.

[Answer]

13. The natural cubic spline $p(t)$ which interpolates the points

$$(1, 3), (2, 1), (3, 6), (4, 6), (5, -3), (6, 1)$$

has coefficient matrix

$$\begin{bmatrix} 2 & -3 & -2 & x & -4 \\ 0 & 6 & y & z & 12 \\ -4 & 2 & 5 & -7 & -4 \\ 3 & 1 & 6 & 6 & -3 \end{bmatrix}$$

(a) Determine the missing values x, y and z.

(b) Calculate the value $p(3.5)$.

[Answer]

14. Find the unique polynomial $p(t)$ of degree (at most) 3 that interpolates the data points $(-1, 0), (0, 2), (1, 2)$ and also satisfies $p'(1) = 1$.

[Answer]

15. Consider $N + 1$ data points $(t_0, y_0), \ldots, (t_N, y_N)$ where $t_i \neq t_j$ $(i \neq j)$. Suppose we want to construct a spline $p(t)$ consisting of N quadratic polynomials

$$p_k(t) = a_k(t - t_{k-1})^2 + b_k(t - t_{k-1}) + c_k \quad , \quad t \in [t_{k-1}, t_k] \quad , \quad k = 1, \ldots, N$$

such that $p(t)$ interpolates the data: $p(t_k) = y_k$, $k = 0, 1, \ldots, N$. Is it always possible to find $p(t)$ such that $p'(t)$ is continuous? Explain.

[Answer]

16. Does there exist a <u>unique</u> function of the form

$$f(t) = c_0 + c_1 e^t + c_2 e^{-t}$$

which satisfies $f(0) = \alpha$, $f'(0) = \beta$ and $f''(0) = \gamma$ for any α, β and γ? Explain.

[Answer]

17. Suppose $p(t)$ is a natural cubic spline with coefficient matrix

$$C = \begin{bmatrix} 1 & -2 & 1 & a_4 & -5 & 6 \\ 0 & 3 & -3 & b_4 & b_5 & -18 \\ 0 & 3 & 3 & 0 & -3 & c_6 \\ 2 & 3 & 7 & 8 & 7 & -4 \end{bmatrix}$$

such that $p(t)$ interpolates $(t_0, y_0), \ldots, (t_6, y_6)$ where

$$t_0 = 0 \ , \ t_1 = 1 \ , \ t_2 = 2 \ , \ t_3 = 3 \ , \ t_4 = 4 \ , \ t_5 = 5 \ , \ t_6 = 6$$

and

$$p_k(t) = a_k(t - t_{k-1})^3 + b_k(t - t_{k-1})^2 + c_k(t - t_{k-1}) + d_k$$

Determine the coefficients a_4, b_4, b_5, c_6.

[Answer]

18. Find all polynomials $p(t)$ of degree 3 (or less) such that

$$p(-1) = 1 \ , \ p(1) = 2 \ , \ p'(-1) = p'(1) = 0$$

[Answer]

19. **True** or **False**: If A is the Vandermonde matrix for the data

$$(-2,1),(-1,2),(0,1),(1,2),(1,1)$$

then $\det(A) = 0$.

[Answer]

20. Find the unique function of the form

$$f(t) = c_0 + c_1 e^t + c_2 e^{-t}$$

which satisfies $f(0) = \alpha$, $f'(0) = \beta$ and $f''(0) = \gamma$. (The answer depends on α, β and γ.)

[Answer]

21. Suppose $p(t)$ is a natural cubic spline with coefficient matrix

$$C = \begin{bmatrix} 1 & -2 & 1 & -1 & -5 & 6 \\ 0 & 3 & -3 & 0 & -3 & -18 \\ 0 & 3 & c_3 & c_4 & -3 & -24 \\ 2 & 3 & d_3 & d_4 & 7 & -4 \end{bmatrix}$$

such that $p(t)$ interpolates $(t_0, y_0), \dots, (t_6, y_6)$ where

$$t_0 = 0 \ , \ t_1 = 1 \ , \ t_2 = 2 \ , \ t_3 = 3 \ , \ t_4 = 4 \ , \ t_5 = 5 \ , \ t_6 = 6$$

and

$$p_k(t) = a_k(t - t_{k-1})^3 + b_k(t - t_{k-1})^2 + c_k(t - t_{k-1}) + d_k$$

Determine the coefficients c_3, d_3, c_4, d_4.

[Answer]

22. Find the unique polynomial $p(t)$ of degree (at most) 3 which satisfies

$$p(1) = 1 \qquad p'(-1) = 1 \qquad p''(1) = 6 \qquad p''(-1) = -6$$

[Answer]

23. **True** or **False**: Consider points $(t_0, y_0), \dots, (t_N, y_N)$ such that $t_0 < t_1 < \cdots < t_N$. There exists a unique function $p(t)$ defined piecewise by quadratic polynomials $p_1(t), \dots, p_N(t)$ such that:

(a) $p_k(t)$ is defined on $[t_{k-1}, t_k]$ for each $k = 1, \dots, N$

(b) $p(t_i) = y_i$ for each $i = 0, \dots, N$

(c) $p(t)$ and $p'(t)$ are continuous on $[t_0, t_N]$

(d) $p'(t_0) = p'(t_N) = 0$.

[Answer]

24. Find the unique cubic polynomial $p(t)$ which satisfies

$$p(-1) = -1 \ , \quad p(0) = 1 \ , \quad p(1) = 1 \ , \quad p'(1) = 5$$

[Answer]

25. The matrix

$$C = \begin{bmatrix} -1 & 2 & 3 & -4 \\ 0 & -3 & 3 & 12 \\ 3 & 0 & 0 & 15 \\ -1 & 1 & 0 & 6 \end{bmatrix}$$

is the coefficient matrix of a natural cubic spline $p(t)$ for 5 data points where

$$t_0 = 0 \ , \quad t_1 = 1 \ , \quad t_2 = 2 \ , \quad t_3 = 3 \ , \quad t_4 = 4$$

Compute $\int_0^4 p(t)dt$.

[Answer]

26. Suppose we have four data points $(0,2), (1,3), (2,2), (3,5)$ and we want to interpolate the data using a polynomial of the form

$$q(t) = d_0 + d_1 t + d_2 t^2 + d_3 t^3 + d_4 t^4,$$

such that $q(t)$ also satisfies $q''''(0) = 0$ (the fourth derivative of $q(t)$ at $t = 0$ is 0).

(a) Set up (but do **not** solve) a linear system $A\boldsymbol{x} = \boldsymbol{b}$ where the solution is the vector of coefficients

$$\boldsymbol{x} = \begin{bmatrix} d_0 \\ d_1 \\ d_2 \\ d_3 \\ d_4 \end{bmatrix}$$

(b) Explain why the system has a unique solution.

[Answer]

27. Consider the data points $(0,2), (1,1)$. We want to find a polynomial of degree at most 3

$$p(t) = c_0 + c_1 t + c_2 t^2 + c_3 t^3$$

such that $p(t)$ interpolates these data points and it also satisfies $p'(0) = 1$ and $p'(1) = -2$.

(a) Write the corresponding system of linear equations in c_0, c_1, c_2, c_3.

(b) Solve this system to find the values of c_0, c_1, c_2, c_3.

(c) Calculate $p(2)$.

[Answer]

28. Consider the data points $(0, 1)$, $(1, 2)$, and $(2, 1)$.

    (a) Find the degree 2 polynomial $p(t)$ that interpolates these points.

    (b) Let $f(t) = 1 + \sin\left(\frac{\pi t}{2}\right)$ be the underlying function that generated the data points. Compute the approximation error $|f(1.5) - p(1.5)|$.

[Answer]

29. Consider the data points: $(0, 1)$, $(1, 2)$, and $(2, 1)$. We now want to fit a cubic spline to these data points, that is, to the data points $(0, 1), (1, 2), (2, 1)$. Determine the missing coefficients, a_2, b_2, c_1, c_2, d_2 in the coefficient matrix of the *natural* cubic spline

$$C = \begin{bmatrix} -0.5 & a_2 \\ 0 & b_2 \\ c_1 & c_2 \\ d_1 & 2 \end{bmatrix}$$

[Answer]

Orthogonality

Subspaces

1. Determine whether or not the set

$$U = \left\{ \begin{bmatrix} a \\ b \\ c \end{bmatrix} \in \mathbb{R}^3 : abc = 0 \right\}$$

is a subspace of $\mathbb{R}^3$. Justify your answer.

[Answer]

2. Determine whether $\operatorname{span}\{\boldsymbol{u}_1, \boldsymbol{u}_2\} = \operatorname{span}\{\boldsymbol{u}_3, \boldsymbol{u}_4\}$ where

$$\boldsymbol{u}_1 = \begin{bmatrix} 2 \\ -3 \\ 1 \\ -1 \end{bmatrix} \qquad \boldsymbol{u}_2 = \begin{bmatrix} -5 \\ 1 \\ 2 \\ -2 \end{bmatrix} \qquad \boldsymbol{u}_3 = \begin{bmatrix} -1 \\ -5 \\ 4 \\ -4 \end{bmatrix} \qquad \boldsymbol{u}_4 = \begin{bmatrix} 3 \\ -11 \\ 6 \\ -10 \end{bmatrix}$$

[Answer]

3. Determine all values c such that the vectors

$$\boldsymbol{u}_1 = \begin{bmatrix} -1 \\ -2 \\ 5 \end{bmatrix} \qquad \boldsymbol{u}_2 = \begin{bmatrix} 2 \\ 3 \\ -6 \end{bmatrix} \qquad \boldsymbol{u}_3 = \begin{bmatrix} 4 \\ 5 \\ c-1 \end{bmatrix}$$

are linearly independent.

[Answer]

4. Determine all values c such that the vectors

$$\boldsymbol{u}_1 = \begin{bmatrix} -1 \\ -2 \\ 5 \end{bmatrix} \qquad \boldsymbol{u}_2 = \begin{bmatrix} 2 \\ 3 \\ -6 \end{bmatrix} \qquad \boldsymbol{u}_3 = \begin{bmatrix} 4 \\ c \\ -4 \end{bmatrix}$$

are linearly independent.

[Answer]

5. **True** or **False**: If $A^3 = 0$ then $R(A) \subseteq N(A)$.

[Answer]

6. **True** or **False**: Let A and B be $m \times n$ matrices. Then the set

$$U = \{\boldsymbol{x} \in \mathbb{R}^n : A\boldsymbol{x} = B\boldsymbol{x}\}$$

is a subspace of $\mathbb{R}^n$.

[Answer]

7. **True** or **False**: If $\{\boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{u}_3\}$ is a basis of a subspace $U \subseteq \mathbb{R}^n$ then

$$\{\ \boldsymbol{u}_1 + \boldsymbol{u}_2 + \boldsymbol{u}_3\ ,\ \boldsymbol{u}_1 + \boldsymbol{u}_3\ ,\ \boldsymbol{u}_1 - \boldsymbol{u}_2 + \boldsymbol{u}_3\ \}$$

is also a basis of U.

[Answer]

8. **True** or **False**: Suppose A_1 and A_2 are $m \times n$ matrices such that $A_1 = LU_1$ and $A_2 = LU_2$. In other words, the unit lower triangular matrix L is the same in both LU decompositions (however the upper triangular matrices U_1 and U_2 are different). If $\text{rank}(A_1) \leq \text{rank}(A_2)$ then $R(A_1) \subseteq R(A_2)$.

[Answer]

9. Suppose $A = LU$ where L is a unit lower triangular matrix and

$$U = \begin{bmatrix} 2 & 1 & 0 & -3 & -2 & 3 \\ 0 & 0 & 1 & -1 & 4 & 1 \\ 0 & 0 & 0 & 0 & 5 & 6 \\ 0 & 0 & 0 & 0 & 0 & -1 \end{bmatrix}$$

Let $\boldsymbol{a}_1, \boldsymbol{a}_2, \boldsymbol{a}_3, \boldsymbol{a}_4, \boldsymbol{a}_5, \boldsymbol{a}_6 \in \mathbb{R}^4$ be the columns of A

$$A = \begin{bmatrix} \boldsymbol{a}_1 & \boldsymbol{a}_2 & \boldsymbol{a}_3 & \boldsymbol{a}_4 & \boldsymbol{a}_5 & \boldsymbol{a}_6 \end{bmatrix}$$

Determine if the statement is **True** or **False**:

(a) The dimension of span$\{\boldsymbol{a}_1, \boldsymbol{a}_2, \boldsymbol{a}_3, \boldsymbol{a}_4\}$ is 2.

(b) The vectors $\boldsymbol{a}_3, \boldsymbol{a}_4, \boldsymbol{a}_5$ are linearly independent.

(c) $\dim(N(A)) = 3$.

(d) The set $\{\boldsymbol{a}_2, \boldsymbol{a}_4, \boldsymbol{a}_5, \boldsymbol{a}_6\}$ is a basis of $R(A)$.

[Answer]

10. Consider the matrix

$$A = \begin{bmatrix} -1 & 1 & 0 & -6 & 2 \\ -1 & -1 & 3 & -8 & 3 \\ 3 & 3 & -9 & 29 & -6 \\ 0 & 12 & -18 & 7 & -9 \end{bmatrix}$$

(a) Find the LU decomposition of A.

(b) Find a basis for $R(A)$.

[Answer]

11. **True** or **False**: If $A^2 = 0$ then $R(A) \subseteq N(A)$.

[Answer]

12. **True** or **False**: If $\{\boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{u}_3\}$ is a basis of a subspace $U \subseteq \mathbb{R}^n$ then

$$\{\ \boldsymbol{u}_1 + \boldsymbol{u}_2\ ,\ \boldsymbol{u}_2 + \boldsymbol{u}_3\ ,\ \boldsymbol{u}_1 + \boldsymbol{u}_3\ \}$$

is also a basis of U.

[Answer]

13. **True** or **False**: Suppose A_1 and A_2 are $m \times n$ matrices such that $A_1 = LU_1$ and $A_2 = LU_2$. In other words, the unit lower triangular matrix L is the same in both LU decompositions (however the upper triangular matrices U_1 and U_2 are different). If $\text{rank}(A_1) \geq \text{rank}(A_2)$ then $R(A_1) \subseteq R(A_2)$.

[Answer]

14. Consider the matrix

$$A = \begin{bmatrix} 2 & -4 & 5 & -6 & -3 \\ -2 & 1 & -11 & 8 & -1 \\ -6 & 6 & -27 & 22 & -2 \\ -10 & 23 & -19 & 28 & 31 \end{bmatrix}$$

(a) Find the LU decomposition of A.

(b) Find a basis for $R(A)$.

[Answer]

15. **True** or **False**: Consider the set

$$U = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} : x + y = z \right\}$$

Then U is a subspace of $\mathbb{R}^3$.

[Answer]

16. **True** or **False**: If A and B are $n \times n$ matrices, then $N(B) \subseteq N(AB)$.

[Answer]

17. Let $\{\boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{u}_3\}$ be a basis of a subspace $U \subseteq \mathbb{R}^n$. Is the set

$$\{\boldsymbol{u}_1, \boldsymbol{u}_1 + \boldsymbol{u}_2, \boldsymbol{u}_1 + \boldsymbol{u}_2 + \boldsymbol{u}_3\}$$

also a basis of U? Explain.

[Answer]

18. Consider the matrix

$$A = \begin{bmatrix} 1 & 3 & 4 & -1 & -2 \\ -3 & -4 & -9 & 4 & 8 \\ 4 & 2 & 10 & 0 & -7 \end{bmatrix}$$

(a) Compute the LU decomposition of A.

(b) Determine $\dim(N(A))$ and $\dim(R(A))$.

[Answer]

19. Find all values of x such that the vector $\boldsymbol{u}$ lies in the plane spanned by the vectors $\boldsymbol{v}$ and $\boldsymbol{w}$ where

$$\boldsymbol{u} = \begin{bmatrix} x \\ 2 \\ -5 \end{bmatrix} \qquad \boldsymbol{v} = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} \qquad \boldsymbol{w} = \begin{bmatrix} 4 \\ 1 \\ -2 \end{bmatrix}$$

[Answer]

20. **True** or **False**: If A and B are $n \times n$ matrices, then $R(A) \subseteq R(AB)$.

[Answer]

21. Consider the matrix

$$A = \begin{bmatrix} 1 & 3 & 4 & -1 & -2 \\ -3 & -4 & -9 & 4 & 8 \\ 4 & 2 & 10 & -6 & -12 \end{bmatrix}$$

(a) Compute the LU decomposition of A.

(b) Determine $\dim(N(A))$ and $\dim(R(A))$.

[Answer]

22. **True** or **False**: If A and B are $n \times n$ matrices, then

$$\dim(R(A+B)) \leq \dim(R(A)) + \dim(R(B))$$

[Answer]

23. **True** or **False**: The subset $U = \left\{ \begin{bmatrix} a \\ b \end{bmatrix} : a = b^2 \right\}$ is a subspace of $\mathbb{R}^2$.

[Answer]

24. Consider the vectors

$$\boldsymbol{u}_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \qquad \boldsymbol{u}_2 = \begin{bmatrix} 2 \\ 3 \\ -1 \end{bmatrix} \qquad \boldsymbol{v}_1 = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} \qquad \boldsymbol{v}_2 = \begin{bmatrix} 4 \\ 1 \\ 3 \end{bmatrix}$$

Determine whether $\text{span}\{\boldsymbol{u}_1, \boldsymbol{u}_2\} = \text{span}\{\boldsymbol{v}_1, \boldsymbol{v}_2\}$.

[Answer]

25. **True** or **False**: If A and B are $n \times n$ matrices, then

$$\dim(N(A+B)) = \dim(N(A)) + \dim(N(B))$$

[Answer]

26. Suppose A is a 4×5 matrix such that $\dim(N(A)) = 3$. Let $\boldsymbol{x}_1, \boldsymbol{x}_2, \boldsymbol{x}_3 \in \mathbb{R}^5$ be any vectors which are **not** in the nullspace of A. Are the vectors $A\boldsymbol{x}_1, A\boldsymbol{x}_2, A\boldsymbol{x}_3 \in \mathbb{R}^4$ linearly independent? Explain.

[Answer]

27. **True** or **False**: There exist (nonzero) vectors $\boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{u}_3, \boldsymbol{u}_4 \in \mathbb{R}^3$ such that

$$\langle \boldsymbol{u}_1, \boldsymbol{u}_2 \rangle = \langle \boldsymbol{u}_2, \boldsymbol{u}_3 \rangle = \langle \boldsymbol{u}_3, \boldsymbol{u}_4 \rangle = \langle \boldsymbol{u}_4, \boldsymbol{u}_1 \rangle = 0$$

[Answer]

28. **True** or **False**: If A is a $m \times n$ matrix, then $N(A^T A) = N(A)$.

[Answer]

29. **True** or **False**: If A is a $m \times n$ matrix, then $R(A^T A) = R(A^T)$.

[Answer]

30. Let A be an $n \times n$ matrix such that $A^T = -A$ and consider the set

$$U = \{\boldsymbol{x} \in \mathbb{R}^n : \boldsymbol{x}^T A \boldsymbol{x} = 0\}$$

Is U a subspace of $\mathbb{R}^n$?

[Answer]

31. Consider the matrix

$$A = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & -2 & 3 & 0 \\ 0 & 1 & -1 & -1 \end{bmatrix}$$

(a) (4 marks) Find the LU decomposition of A.

(b) (2 marks) Find a basis for $N(A)$.

[Answer]

32. **True** or **False**: Let A and B be $n \times n$ matrices. If $AB = 0$ then

$$\text{rank}(A) + \text{rank}(B) \leq n$$

[Answer]

33. Let $n > 2$ be an integer, let $\boldsymbol{u}, \boldsymbol{v} \in \mathbb{R}^n$ be linearly independent vectors and let $A = \boldsymbol{u}\boldsymbol{u}^T + \boldsymbol{v}\boldsymbol{v}^T$. Determine $\dim(N(A))$ and $\dim(R(A))$.

[Answer]

34. Let

$$\boldsymbol{x}_1 = \begin{bmatrix} 1 \\ 4 \\ 5 \\ 1 \end{bmatrix} \qquad \boldsymbol{x}_2 = \begin{bmatrix} 4 \\ 1 \\ 8 \\ 2 \end{bmatrix} \qquad \boldsymbol{x}_3 = \begin{bmatrix} 3 \\ 1 \\ 3 \\ 1 \end{bmatrix} \qquad \boldsymbol{x}_4 = \begin{bmatrix} 5 \\ 6 \\ 1 \\ 1 \end{bmatrix}$$

and

$$A = \begin{bmatrix} \boldsymbol{x}_1 & \boldsymbol{x}_2 & \boldsymbol{x}_3 & \boldsymbol{x}_4 \end{bmatrix}$$

In other words, the vectors $\boldsymbol{x}_1, \boldsymbol{x}_2, \boldsymbol{x}_3, \boldsymbol{x}_4$ are the columns of A.

(a) Is the set $\{\boldsymbol{x}_1, \boldsymbol{x}_2, \boldsymbol{x}_3, \boldsymbol{x}_4\}$ linearly independent?

(b) Does the set $\{\boldsymbol{x}_1, \boldsymbol{x}_2, \boldsymbol{x}_3, \boldsymbol{x}_4\}$ span $\mathbb{R}^4$?

(c) What is $\dim N(A)$?

[Answer]

35. Let

$$\boldsymbol{x}_1 = \begin{bmatrix}1\\2\\1\\4\end{bmatrix} \qquad \boldsymbol{x}_2 = \begin{bmatrix}1\\7\\2\\3\end{bmatrix} \qquad \boldsymbol{x}_3 = \begin{bmatrix}2\\2\\1\\1\end{bmatrix} \qquad \boldsymbol{x}_4 = \begin{bmatrix}1\\2\\4\\3\end{bmatrix}$$

and

$$A = \begin{bmatrix} \boldsymbol{x}_1 & \boldsymbol{x}_2 & \boldsymbol{x}_3 & \boldsymbol{x}_4 \end{bmatrix}$$

In other words, the vectors $\boldsymbol{x}_1, \boldsymbol{x}_2, \boldsymbol{x}_3, \boldsymbol{x}_4$ are the columns of A.

(a) Find a basis for $R(A)$.

(b) Find a basis for $\text{span}\{\boldsymbol{x}_1, \boldsymbol{x}_2, \boldsymbol{x}_3, \boldsymbol{x}_1 + \boldsymbol{x}_2 + \boldsymbol{x}_3\}$.

(c) What is $\dim N(A)$?

[Answer]

Orthogonal Subspaces

1. **True** or **False**: Let $U \subset \mathbb{R}^5$ be a subspace such that $\dim(U) = 2$. There exists a subspace $V \subset \mathbb{R}^5$ such that $\dim(V) = 4$ and V is orthogonal to U.

[Answer]

2. **True** or **False**: If U_1 and U_2 are orthogonal subspaces of $\mathbb{R}^n$, then the orthogonal complements $U_1^\perp$ and $U_2^\perp$ are orthogonal. In other words, if $U_1 \perp U_2$ then $U_1^\perp \perp U_2^\perp$.

[Answer]

3. **True** or **False**: There exist (nonzero) vectors $\boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{u}_3, \boldsymbol{u}_4 \in \mathbb{R}^3$ such that

$$\langle \boldsymbol{u}_1, \boldsymbol{u}_2 \rangle = \langle \boldsymbol{u}_2, \boldsymbol{u}_3 \rangle = \langle \boldsymbol{u}_3, \boldsymbol{u}_4 \rangle = \langle \boldsymbol{u}_4, \boldsymbol{u}_1 \rangle = 0$$

[Answer]

4. **True** or **False**: Let $U \subset \mathbb{R}^5$ such that $\dim(U) = 2$. There exists a unique subspace $V \subset \mathbb{R}^5$ such that $\dim(V) = 2$ and $U \perp V$.

[Answer]

5. **True** or **False**: If $U, V \subset \mathbb{R}^n$ such that $U \perp V$ then $\dim(U) + \dim(V) = n$.

[Answer]

6. Find an orthogonal basis for the subspace $U = \text{span}\left\{ \begin{bmatrix}1\\2\\0\end{bmatrix}, \begin{bmatrix}2\\0\\-1\end{bmatrix} \right\}$.

[Answer]

7. **True** or **False**: If A is a real matrix with orthogonal columns, then A^TA is a diagonal matrix.

[Answer]

8. Let $U = \text{span}\{\boldsymbol{v}_1, \boldsymbol{v}_2\} \subseteq \mathbb{R}^4$ where

$$\boldsymbol{v}_1 = \begin{bmatrix} 2 \\ 1 \\ 3 \\ -1 \end{bmatrix} \qquad \boldsymbol{v}_2 = \begin{bmatrix} 1 \\ -2 \\ 0 \\ 2 \end{bmatrix}$$

   (a) Show that $\boldsymbol{v}_1, \boldsymbol{v}_2$ are linearly independent and hence $\dim(U) = 2$.
   (b) Find a basis for $U^\perp$.

[Answer]

Fundamental Subspaces of a Matrix

1. Let $a, b \in \mathbb{R}$ such that $a \neq b$ and consider the matrix

$$A = \begin{bmatrix} a & b & a & b & a & b \\ b & a & b & a & b & a \\ a & b & a & b & a & b \\ b & a & b & a & b & a \\ a & b & a & b & a & b \end{bmatrix}$$

   Determine the dimension of $N(A)^\perp$.

[Answer]

2. Let $a, b \in \mathbb{R}$ such that $a \neq b$ and consider the matrix

$$A = \begin{bmatrix} a & b & a & b & a & b \\ a & b & a & b & a & b \\ b & a & b & a & b & a \\ b & a & b & a & b & a \\ a & b & a & b & a & b \end{bmatrix}$$

   Determine the dimension of $R(A)^\perp$.

[Answer]

3. Consider a matrix A with LU decomposition $A = LU$ where

$$L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 2 & 2 & 0 & 1 \end{bmatrix} \qquad U = \begin{bmatrix} 1 & 3 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Determine the dimension of $N(A)^\perp$.

[Answer]

4. Consider a matrix A with LU decomposition $A = LU$ where

$$L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 2 & 2 & 0 & 1 \end{bmatrix} \qquad U = \begin{bmatrix} 1 & 3 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Determine the dimension of $N(A)^\perp$.

[Answer]

5. Consider the matrix

$$A = \begin{bmatrix} -1 & -2 & -2 & -1 \\ 4 & 10 & 11 & 2 \\ -3 & -2 & 0 & -7 \end{bmatrix}$$

   (a) Compute the LU decomposition of A.
   (b) Determine the dimension of $N(A^T)$.
   (c) Find a basis of $N(A)$.

[Answer]

6. **True** or **False**: If A is a symmetric matrix then $N(A)^\perp = R(A)$.

[Answer]

7. Determine the dimension of $R(A)^\perp$ where

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 3 & 4 & 5 & 6 \\ 3 & 4 & 5 & 6 & 7 \\ 4 & 5 & 6 & 7 & 8 \\ 5 & 6 & 7 & 8 & 9 \end{bmatrix}$$

[Answer]

8. Consider the matrix

$$A = \begin{bmatrix} 1 & 4 & -1 & 0 \\ -1 & -5 & 1 & 3 \\ 2 & 8 & 0 & -2 \\ 1 & 3 & -2 & 4 \end{bmatrix}$$

   (a) Find the LU decomposition of A.
   (b) Find a basis of $R(A)^\perp$.

[Answer]

9. Find the dimension of $R(A)^{\perp}$ for the matrix

$$A = \begin{bmatrix} 1 & 2 & 3 & 2 & 1 \\ 2 & 1 & 2 & 3 & 2 \\ 3 & 2 & 1 & 2 & 3 \\ 2 & 3 & 2 & 1 & 2 \\ 1 & 2 & 3 & 2 & 1 \end{bmatrix}$$

[Answer]

10. Consider a 6×3 matrix A with $\text{rank}(A) = 2$. Determine the dimensions of the subspaces $N(A)$, $R(A)$, $N(A^T)$ and $R(A^T)$.

[Answer]

11. **True** or **False**: There exists a matrix A such that $R(A) = N(A)$.

[Answer]

Orthogonal Projection

1. Find the orthogonal projection matrix P which projects onto $U = \text{span}\{\boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{u}_3\}$ where

$$\boldsymbol{u}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \qquad \boldsymbol{u}_2 = \begin{bmatrix} 1 \\ -1 \\ 1 \\ 1 \end{bmatrix} \qquad \boldsymbol{u}_3 = \begin{bmatrix} 1 \\ 0 \\ 2 \\ 1 \end{bmatrix}$$

[Answer]

2. Find the orthogonal projection matrix P which projects onto $U = \text{span}\{\boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{u}_3\}$ where

$$\boldsymbol{u}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \qquad \boldsymbol{u}_2 = \begin{bmatrix} 1 \\ -1 \\ 1 \\ 1 \end{bmatrix} \qquad \boldsymbol{u}_3 = \begin{bmatrix} -2 \\ 0 \\ 1 \\ 1 \end{bmatrix}$$

[Answer]

3. Find the shortest distance from $\boldsymbol{x} = \begin{bmatrix} 5 \\ 3 \\ 7 \end{bmatrix}$ to the plane in $\mathbb{R}^3$ given by

$$U = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \in \mathbb{R}^3 : 2x - y + 3z = 0 \right\}.$$

[Answer]

4. Find the shortest distance from $\boldsymbol{x} = \begin{bmatrix} 5 \\ 3 \\ 7 \end{bmatrix}$ to the plane in $\mathbb{R}^3$ given by

$$U = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \in \mathbb{R}^3 : x - 3y + z = 0 \right\}.$$

[Answer]

5. **True** or **False**: Let U_1 and U_2 be subspaces of $\mathbb{R}^n$. Let P_1 be the projection onto U_1 and let P_2 be the projection onto U_2. If $P_1 P_2 = 0$ then U_1 and U_2 are orthogonal subspaces.

[Answer]

6. Find the shortest distance from $\boldsymbol{x}$ to $U = \text{span}\{\boldsymbol{u}_1, \boldsymbol{u}_2\} \subseteq \mathbb{R}^4$ where

$$\boldsymbol{u}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \qquad \boldsymbol{u}_2 = \begin{bmatrix} 1 \\ 1 \\ 2 \\ 0 \end{bmatrix} \qquad \boldsymbol{x} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}$$

[Answer]

7. **True** or **False**: Let P be the projection matrix onto a subspace $U \subset \mathbb{R}^6$ with $\dim(U) = 4$. Then the rank of the matrix $I - P$ is 4.

[Answer]

8. If P is a 5×5 projection matrix such that $\text{rank}(P) = 2$ then determine the dimension of the nullspace $N(P)$.

[Answer]

9. Find the projection matrix P which projects onto $U = \text{span}\{\boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{u}_3\}$ where

$$\boldsymbol{u}_1 = \begin{bmatrix} -1 \\ 1 \\ 1 \\ 2 \end{bmatrix} \qquad \boldsymbol{u}_2 = \begin{bmatrix} 1 \\ 0 \\ 2 \\ -1 \end{bmatrix} \qquad \boldsymbol{u}_3 = \begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \end{bmatrix}$$

[Answer]

10. **True** or **False**: The matrix

$$P = \frac{1}{\sqrt{14}} \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 3 & 6 & 9 \end{bmatrix}$$

is the projection matrix onto a subspace U such that $\dim(U) = 2$.

[Answer]

11. Find the shortest distance from $\boldsymbol{x}$ to $U = \text{span}\{\boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{u}_3\}$ where

$$\boldsymbol{u}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \qquad \boldsymbol{u}_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 2 \end{bmatrix} \qquad \boldsymbol{u}_3 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix} \qquad \boldsymbol{x} = \begin{bmatrix} -1 \\ 1 \\ 0 \\ 1 \end{bmatrix}$$

[Answer]

12. **True** or **False**: Let P_1 be the projection matrix onto the subspace $U_1 \subset \mathbb{R}^n$ and let P_2 be the projection matrix onto the subspace $U_2 \subset \mathbb{R}^n$. If $P_2P_1 = 0$ then $U_1 \perp U_2$.
[Answer]

13. Determine the values $a_1, a_2, a_3, a_4, a_5, a_6, a_7$ such that the matrix

$$P = \frac{1}{a_7}\begin{bmatrix} 1 & a_1 & 2 \\ 3 & a_2 & a_3 \\ a_4 & a_5 & a_6 \end{bmatrix}$$

is the projection matrix onto a vector $\boldsymbol{u}$.

[Answer]

14. **True** or **False**: Let P_1 be the projection matrix onto the subspace $U_1 \subset \mathbb{R}^n$ and let P_2 be the projection matrix onto the subspace $U_2 \subset \mathbb{R}^n$. If $P_2P_1 = 0$ then $U_1 \subset U_2$.
[Answer]

15. Determine the values $a_1, a_2, a_3, a_4, a_5, a_6, a_7$ such that the matrix

$$P = \frac{1}{a_7}\begin{bmatrix} 1 & a_1 & -1 \\ 4 & a_2 & a_3 \\ a_4 & a_5 & a_6 \end{bmatrix}$$

is the projection matrix onto a vector $\boldsymbol{u}$.

[Answer]

16. **True** or **False**: Let P_1 be the projection matrix onto the subspace $U_1 \subset \mathbb{R}^n$ and let P_2 be the projection matrix onto the subspace $U_2 \subset \mathbb{R}^n$. If $P_2P_1 = 0$ then $U_1 \perp U_2$.
[Answer]

17. **True** or **False**: Let P_1 be the projection matrix onto the subspace $U_1 \subset \mathbb{R}^n$ and let P_2 be the projection matrix onto the subspace $U_2 \subset \mathbb{R}^n$. If $U_1 \subset U_2$ then $P_2P_1 = P_2$.
[Answer]

18. Compute the projection of $\boldsymbol{x}$ onto $N(A^T)$ where

$$\boldsymbol{x} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \qquad A = \begin{bmatrix} 1 & 1 \\ 2 & 1 \\ 0 & 2 \\ 1 & 3 \end{bmatrix}$$

[Answer]

19. Find the orthogonal projection matrix P that projects onto the plane spanned by

$$\boldsymbol{u}_1 = \begin{bmatrix} 1 \\ 3 \\ 0 \end{bmatrix} \qquad \boldsymbol{u}_2 = \begin{bmatrix} 2 \\ 7 \\ 1 \end{bmatrix}$$

[Answer]

20. Consider the matrix A with LU decomposition $A = LU$ where

$$U = \begin{bmatrix} 1 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix}$$

Find the shortest distance from $\boldsymbol{x}$ to $R(A^T)$ for the vector

$$\boldsymbol{x} = \begin{bmatrix} 1 \\ 1 \\ 2 \\ 1 \\ 2 \end{bmatrix}$$

[Answer]

21. **True** or **False**: Let $\boldsymbol{u}, \boldsymbol{v} \in \mathbb{R}^n$ be nonzero vectors such that $\langle \boldsymbol{u}, \boldsymbol{v} \rangle = 0$. Let P_1 be the projection matrix onto $\boldsymbol{u}$ and let P_2 be the projection matrix onto $\boldsymbol{v}$. Then the projection matrix onto $\text{span}\{\boldsymbol{u}, \boldsymbol{v}\}$ is $P_1 + P_2$.

[Answer]

22. Find the shortest distance from $\boldsymbol{b}$ to $R(A)$ where

$$A = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix} \qquad \boldsymbol{b} = \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \\ 1 \end{bmatrix}$$

[Answer]

23. **True** or **False**: Let P_1 be the projection matrix onto $U \subset \mathbb{R}^n$ and let P_2 be the projection matrix onto $U^\perp \subset \mathbb{R}^n$. Then $P_1 + P_2 = 0$.

[Answer]

24. Find the projection matrix P that projects onto $U = \text{span}\{\boldsymbol{u}_1, \boldsymbol{u}_2\}$ where

$$\boldsymbol{u}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \qquad \boldsymbol{u}_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 2 \end{bmatrix}$$

[Answer]

25. **True** or **False**: Let A, B be $n \times n$ matrices that are orthogonal projections. If $R(A) \perp R(B)$, then $A + B$ is an orthogonal projection.

[Answer]

26. Let $\{\boldsymbol{x}_1, \boldsymbol{x}_2\}$ be a basis of the subspace $U \subseteq \mathbb{R}^3$ where

$$\boldsymbol{x}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \qquad \boldsymbol{x}_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

(a) Find a 3×2 matrix A such that $U = N(A^T)^\perp$.

(b) Find a basis for $U^\perp$.

(c) Construct an orthogonal projection matrix which projects onto U.

[Answer]

27. **True** or **False**: Let A be a nonzero $n \times n$ projection matrix. If $\boldsymbol{y}$ is a nonzero vector in $R(A)$, then $\boldsymbol{y}$ is not in $N(A)$.

[Answer]

28. Let $U = \text{span}\{\boldsymbol{x}\} \subset \mathbb{R}^3$ where

$$\boldsymbol{x} = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$$

(a) Construct the projection matrix which projects onto U.

(b) Find a projection matrix P such that $N(P) = U$. (Hint: such a matrix P must satisfy $R(P^T) = N(P)^\perp = U^\perp$.)

(c) Find a basis for $U^\perp$.

[Answer]

29. Let P be the projection matrix onto $\text{span}\{\boldsymbol{u}, \boldsymbol{v}\}$ where

$$\boldsymbol{u} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \qquad \boldsymbol{v} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

Compute $(I - P)^T \big(P^4 \boldsymbol{x} - (I - P)^2 \boldsymbol{x}\big)$ where

$$\boldsymbol{x} = \begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix}$$

[Answer]

30. Consider the subspace $W = \text{span}\{\boldsymbol{w}_1, \boldsymbol{w}_2, \boldsymbol{w}_3\} \subset \mathbb{R}^4$ where

$$\boldsymbol{w}_1 = \begin{bmatrix} 1 \\ 2 \\ 2 \\ 0 \end{bmatrix} \quad \boldsymbol{w}_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 2 \end{bmatrix} \quad \boldsymbol{w}_3 = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

(a) Show that $\boldsymbol{w}_1, \boldsymbol{w}_2, \boldsymbol{w}_3$ are linearly independent.

(b) Use Gram–Schmidt to produce an orthonormal basis $\{\boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{u}_3\}$ of W.

(c) Compute the projection matrix P that projects onto W.

[Answer]

QR Decomposition

1. **True** or **False**: If A is an $n \times n$ symmetric matrix such that $A^2 = I$, then A is orthogonal.

[Answer]

2. Suppose $A = Q_1 R_1$ where

$$Q_1 = \frac{1}{2} \begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & -1 & -1 \\ 1 & 1 & -1 \end{bmatrix} \qquad R_1 = \begin{bmatrix} -1 & 1 & 2 \\ 0 & -2 & -1 \\ 0 & 0 & 1 \end{bmatrix}$$

Compute the projection of $\boldsymbol{v}$ onto $R(A)^\perp$ for

$$\boldsymbol{v} = \begin{bmatrix} 1 \\ 2 \\ -1 \\ 3 \end{bmatrix}$$

[Answer]

3. **True** or **False**: If A is an $n \times n$ orthogonal matrix such that $A^2 = I$, then A is symmetric.

[Answer]

4. Suppose $A = Q_1 R_1$ where

$$Q_1 = \frac{1}{2} \begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & -1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{bmatrix} \qquad R_1 = \begin{bmatrix} -1 & 1 & 2 \\ 0 & -2 & -1 \\ 0 & 0 & 1 \end{bmatrix}$$

Compute the projection of $\boldsymbol{v}$ onto $R(A)^{\perp}$ for

$$\boldsymbol{v} = \begin{bmatrix} 1 \\ -2 \\ -3 \\ 1 \end{bmatrix}$$

[Answer]

5. Determine the values of a, b and c such that

$$Q = \begin{bmatrix} 1/\sqrt{18} & a & 2/3 \\ 1/\sqrt{18} & 1/\sqrt{2} & b \\ -4/\sqrt{18} & 0 & c \end{bmatrix}$$

is an orthogonal matrix.

[Answer]

6. Consider the matrix $A = Q_1 R_1$ where

$$Q_1 = \frac{1}{2}\begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & -1 & -1 \\ 1 & 1 & -1 \end{bmatrix} \qquad R_1 = \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Compute the orthogonal projection of $\boldsymbol{v}$ onto $N(A^T)$ where

$$\boldsymbol{v} = \begin{bmatrix} 2 \\ 1 \\ -1 \\ 3 \end{bmatrix}$$

[Answer]

7. Suppose $A = Q_1 R_1$ where

$$Q_1 = \frac{1}{2}\begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & -1 & -1 \\ 1 & 1 & -1 \end{bmatrix} \qquad R_1 = \begin{bmatrix} -1 & 1 & 2 \\ 0 & -2 & -1 \\ 0 & 0 & 1 \end{bmatrix}$$

Find the least squares approximation $A\boldsymbol{x} \approx \boldsymbol{b}$ for

$$\boldsymbol{b} = \begin{bmatrix} 2 \\ 1 \\ -1 \\ 1 \end{bmatrix}$$

[Answer]

8. Determine values a, b and c such that

$$Q = \begin{bmatrix} 1/\sqrt{2} & a & 1/\sqrt{18} \\ 0 & 1/3 & b \\ 1/\sqrt{2} & 2/3 & c \end{bmatrix}$$

is an orthogonal matrix.

[Answer]

9. Consider the matrix $A = Q_1 R_1$ where

$$Q_1 = \frac{1}{2}\begin{bmatrix} 1 & 1 & 1 \\ -1 & -1 & 1 \\ 1 & -1 & -1 \\ -1 & 1 & -1 \end{bmatrix} \qquad R_1 = \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Compute the orthogonal projection of $\boldsymbol{v}$ onto $N(A^T)$ where

$$\boldsymbol{v} = \begin{bmatrix} 1 \\ 2 \\ -1 \\ 3 \end{bmatrix}$$

[Answer]

10. **True** or **False**: Let A be an $m \times n$ matrix with $\text{rank}(A) = n$. Let $A = QR$ be the QR decomposition with

$$Q = [\ Q_1\ Q_2\ ] \qquad R = \begin{bmatrix} R_1 \\ \mathbf{0} \end{bmatrix}$$

where $A = Q_1 R_1$ is the thin QR decomposition. The projection of $\boldsymbol{x} \in \mathbb{R}^m$ onto $R(A)^\perp$ is equal to $Q_2 Q_2^T \boldsymbol{x}$.

[Answer]

11. Find the <u>thin</u> QR decomposition of

$$A = \begin{bmatrix} -1 & 1 & 1 \\ 1 & 2 & 0 \\ 1 & 0 & -1 \\ 1 & -1 & 1 \end{bmatrix}$$

[Answer]

12. Suppose $A = QR$ is the QR decomposition of A where

$$Q = \frac{1}{\sqrt{2}}\begin{bmatrix} -1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & -1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix} \qquad R = \begin{bmatrix} 2\sqrt{2} & -\sqrt{2} \\ 0 & -2\sqrt{2} \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Determine the shortest distance from $\boldsymbol{x}$ to $R(A)$ where

$$\boldsymbol{x} = \begin{bmatrix} 1 \\ 2 \\ 3 \\ -2 \end{bmatrix}$$

[Answer]

13. **True** or **False**: Let A be a $m \times n$ matrix such that $\operatorname{rank}(A) = n$ and let $A = Q_1 R_1$ be the thin QR decomposition. Then $Q_1^T Q_1 = I$.

[Answer]

14. **True** or **False**: Let A be a $m \times n$ matrix with $\operatorname{rank}(A) = n$ and let $A = QR$ be the QR decomposition of A with $Q = [\, Q_1 \; Q_2 \,]$ where $A = Q_1 R_1$ is the thin QR decomposition. Then $R(Q_2) = N(A^T)$.

[Answer]

15. Let $A = Q_1 R_1$ and $\boldsymbol{x} \in \mathbb{R}^4$ where

$$Q_1 = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ -1 & 1 \\ 1 & 1 \\ 1 & -1 \end{bmatrix} \qquad R_1 = \begin{bmatrix} -4 & 5 \\ 0 & -2 \end{bmatrix} \qquad \boldsymbol{x} = \begin{bmatrix} 2 \\ 1 \\ -1 \\ a \end{bmatrix}$$

where $a \in \mathbb{R}$. Find the value a such that $\boldsymbol{x}$ is nearest to the subspace $R(A)$.

[Answer]

16. **True** or **False**: Let A be a $m \times n$ matrix with $\operatorname{rank}(A) = n$ and let $A = QR$ be the QR decomposition of A with $Q = [\, Q_1 \; Q_2 \,]$ where $A = Q_1 R_1$ is the thin QR decomposition. Then $Q_1^T Q_2 = 0$.

[Answer]

17. **True** or **False**: Let A be a $m \times n$ matrix with $\operatorname{rank}(A) = n$ and let $A = QR$ be the QR decomposition of A with $Q = [\, Q_1 \; Q_2 \,]$ where $A = Q_1 R_1$ is the thin QR decomposition. Then $Q_1 Q_1^T = I - Q_2 Q_2^T$.

[Answer]

18. Compute the <u>thin</u> QR decomposition of the matrix

$$A = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 0 & 1 \\ 2 & -1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

[Answer]

19. Let A be a $m \times n$ matrix with $\text{rank}(A) = n$ and $m > n$. Let $A = Q_1 R_1$ be the thin QR decomposition. Show that $(A^T A)^{-1} A^T = R_1^{-1} Q_1^T$.

[Answer]

20. Compute the thin QR decomposition for the matrix

$$A = \begin{bmatrix} 1 & 0 & 5 \\ 0 & -3 & 4 \\ -2 & 1 & -3 \\ 0 & -5 & 4 \\ 2 & 1 & -1 \end{bmatrix}$$

[Answer]

21. **True** or **False**: Let A be a $m \times n$ matrix such that the columns are orthogonal and let $A = QR$ be the QR decomposition. Then R is a diagonal matrix.

[Answer]

22. Find a <u>thin</u> QR decomposition of

$$A = \begin{bmatrix} 1 & -1 \\ 2 & 0 \\ 2 & 2 \end{bmatrix}$$

[Answer]

23. Let

$$A = \begin{bmatrix} -2 & 0 \\ 2 & -1 \\ 2 & -1 \\ -2 & 0 \end{bmatrix}$$

(a) Compute the thin QR decomposition $A = Q_1 R_1$.

(b) Compute the projection matrix P_1 which projects onto $R(A)$.

(c) Compute the projection matrix P_2 which projects onto $R(A)^\perp$.

[Answer]

24. Let $A = Q_1 R_1$ and $\boldsymbol{x} \in \mathbb{R}^4$ where

$$Q_1 = \begin{bmatrix} 1/2 & -5/6 \\ 1/2 & 1/6 \\ 1/2 & 3/6 \\ 1/2 & 1/6 \end{bmatrix} \qquad R_1 = \begin{bmatrix} 4 & 2 \\ 0 & 3 \end{bmatrix} \qquad \boldsymbol{x} = \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix}$$

(a) Compute $P_1 \boldsymbol{x}$ where P_1 is the projection matrix which projects onto $R(A)$.

(b) Compute $P_2\boldsymbol{x}$ where P_2 is the projection matrix which projects onto $R(A)^\perp$.

[Answer]

Least Squares Approximation

1. Use least squares linear regression to find the linear function $f(t) = c_0 + c_1 t$ that best fits the data points $(0,1),(1,3),(3,4)$ and $(4,3)$.

[Answer]

2. Suppose $A = Q_1 R_1$ where

$$Q_1 = \frac{1}{2}\begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & -1 & -1 \\ 1 & 1 & -1 \end{bmatrix} \qquad R_1 = \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Find the least squares approximation $A\boldsymbol{x} \approx \boldsymbol{b}$ for

$$\boldsymbol{b} = \begin{bmatrix} 1 \\ 3 \\ 2 \\ -1 \end{bmatrix}$$

[Answer]

3. Consider the matrix

$$A = \begin{bmatrix} 1 & -1 & 1 \\ 1 & 0 & 2 \\ 0 & 1 & 1 \\ -1 & -1 & 0 \end{bmatrix}$$

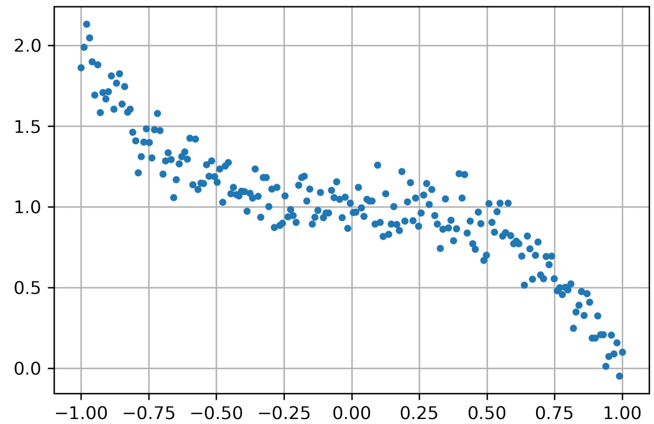
(a) (5 marks) Compute the thin QR decomposition $A = Q_1 R_1$.

(b) (3 marks) Use the thin QR decomposition to find the least squares approximation $A\boldsymbol{x} \approx \boldsymbol{b}$ for the vector

$$\boldsymbol{b} = \begin{bmatrix} 1 \\ 1 \\ 2 \\ 0 \end{bmatrix}$$

[Answer]

4. The figure below shows 200 data points $(t_1, y_1), \ldots, (t_{200}, y_{200})$



Determine (approximately) the least squares approximation $A\boldsymbol{c} \approx \boldsymbol{y}$ where

$$A = \begin{bmatrix} 1 & t_1 & t_1^2 & t_1^3 \\ 1 & t_2 & t_2^2 & t_2^3 \\ \vdots & \vdots & \vdots & \vdots \\ 1 & t_{199} & t_{199}^2 & t_{199}^3 \\ 1 & t_{200} & t_{200}^2 & t_{200}^3 \end{bmatrix} \qquad \boldsymbol{y} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_{199} \\ y_{200} \end{bmatrix}$$

[Answer]

5. Let $A = Q_1 R_1$ be the thin QR decomposition of A, and let $\boldsymbol{b} \in \mathbb{R}^4$ where

$$Q_1 = \frac{1}{2}\begin{bmatrix} 1 & -1 \\ 1 & 1 \\ 1 & 1 \\ 1 & -1 \end{bmatrix} \qquad R_1 = \begin{bmatrix} 2 & 1 \\ 0 & 5 \end{bmatrix} \qquad \boldsymbol{b} = \begin{bmatrix} 1 \\ 2 \\ 0 \\ 2 \end{bmatrix}$$

   (a) Find the projection of $\boldsymbol{b}$ onto $R(A)^\perp$.
   (b) Find the least squares approximation $A\boldsymbol{x} \approx \boldsymbol{b}$.

[Answer]

6. Let $A = Q_1 R_1$ and $\boldsymbol{b} \in \mathbb{R}^4$ where

$$Q_1 = \frac{1}{3}\begin{bmatrix} 1 & -2 & 0 \\ 2 & 0 & -2 \\ 2 & 1 & 2 \\ 0 & 2 & -1 \end{bmatrix} \qquad R_1 = \begin{bmatrix} 3 & 2 & -1 \\ 0 & -1 & 1 \\ 0 & 0 & 3 \end{bmatrix} \qquad \boldsymbol{b} = \begin{bmatrix} 3 \\ 3 \\ -3 \\ 3 \end{bmatrix}$$

Find the least squares approximation $A\boldsymbol{x} \approx \boldsymbol{b}$.

[Answer]

7. Use least squares approximation to find the function of the form $f(t) = c_0 + c_1 t$ which best fits the data $(-2,-1)$, $(-1,0)$, $(0,0)$, $(1,1)$ and $(2,2)$.

[Answer]

8. Setup but do **not** solve a linear system $A\boldsymbol{c} = \boldsymbol{y}$ such that the least squares solution $\boldsymbol{c}$ is the vector of coefficients of the function

$$f(t) = c_0 + c_1 e^{-t} + c_2 t e^{-t} \ , \quad \boldsymbol{c} = \begin{bmatrix} c_0 \\ c_1 \\ c_2 \end{bmatrix}$$

which best fits the data $(0,0)$, $(1,3)$, $(2,2)$, $(3,1)$, $(4,1)$, and $(5,1)$.

[Answer]

9. Let $A = Q_1 R_1$ be the thin QR decomposition of A, and let $\boldsymbol{b} \in \mathbb{R}^4$ where

$$Q_1 = \begin{bmatrix} 1/\sqrt{6} & -1/\sqrt{6} & -1/\sqrt{3} \\ 0 & 2/\sqrt{6} & 0 \\ 2/\sqrt{6} & 0 & 1/\sqrt{3} \\ 1/\sqrt{6} & 1/\sqrt{6} & -1/\sqrt{3} \end{bmatrix} \qquad R_1 = \begin{bmatrix} \sqrt{6} & 2\sqrt{6} & -\sqrt{6} \\ 0 & \sqrt{6} & 2\sqrt{6} \\ 0 & 0 & \sqrt{3} \end{bmatrix} \qquad \boldsymbol{b} = \begin{bmatrix} 1 \\ 2 \\ 0 \\ 2 \end{bmatrix}$$

(a) Find the projection of $\boldsymbol{b}$ onto $R(A)^{\perp}$.

(b) Use the thin QR decomposition of A to find the least squares approximation $A\boldsymbol{x} \approx \boldsymbol{b}$.

[Answer]

10. Calculate the least squares solution $A\boldsymbol{x} \approx \boldsymbol{b}$ where

$$A = \begin{bmatrix} 1 & 2 \\ 1 & -1 \\ 1 & -1 \end{bmatrix} \qquad \boldsymbol{b} = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$$

[Answer]

11. Let $A = Q_1 R_1$ where

$$Q_1 = \begin{bmatrix} 1/3 & -2/3 \\ 2/3 & -1/3 \\ 2/3 & 2/3 \end{bmatrix} \qquad R_1 = \begin{bmatrix} 3 & 4 \\ 0 & 2 \end{bmatrix}$$

Find the least squares approximation $A\boldsymbol{x} \approx \boldsymbol{b}$ where

$$\boldsymbol{b} = \begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix}$$

[Answer]

12. Let $\alpha \in \mathbb{R}$, $\boldsymbol{b} \in \mathbb{R}^4$, let A be a 4×2 matrix with QR decomposition $A = QR$ and suppose $\boldsymbol{x}$ is the least squares solution of $A\boldsymbol{x} \approx \boldsymbol{b}$ where

$$Q = \frac{1}{2}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix} \qquad R = \begin{bmatrix} 1 & \alpha \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \qquad \boldsymbol{b} = \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix} \qquad \boldsymbol{x} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

(a) Compute α.

(b) Compute the residual norm $\|A\boldsymbol{x} - \boldsymbol{b}\|$.

(c) Compute the squared norm of the projection of $\boldsymbol{b}$ onto the column space of A.

[Answer]

13. Let $A = Q_1 R_1$ where

$$Q_1 = \frac{1}{6}\begin{bmatrix} -5 & 3 & -1 \\ 3 & 5 & 1 \\ -1 & 1 & 3 \\ -1 & -1 & 5 \end{bmatrix} \qquad R_1 = \begin{bmatrix} 1 & -1 & 2 \\ 0 & 2 & 3 \\ 0 & 0 & 5 \end{bmatrix} \qquad \boldsymbol{b} = \begin{bmatrix} 6 \\ 12 \\ -12 \\ -6 \end{bmatrix}$$

Setup and solve a linear system $B\boldsymbol{c} = \boldsymbol{y}$ where the solution is the least squares approximation of $A\mathbf{x} \approx \mathbf{b}$.

[Answer]

14. Let $f(t) = c_1 + c_2 t + c_3 t^2$ and consider the data

$$(0, -1), (1, 2), (2, -1), (-1, 2), (-2, 1)$$

Setup and solve a linear system $A\boldsymbol{c} = \boldsymbol{y}$ where the solution $\boldsymbol{c}$ is the vector of coefficients c_1, c_2, c_3 such that $f(t)$ best fits the data.

[Answer]

Eigenvalues and Eigenvectors

Eigenvalues and Eigenvectors

1. Let $\boldsymbol{u}_1, \boldsymbol{u}_2 \in \mathbb{R}^3$ such that $\|\boldsymbol{u}_1\| = a > 0$, $\|\boldsymbol{u}_2\| = b > 0$, and $\langle \boldsymbol{u}_1, \boldsymbol{u}_2 \rangle = 0$. Let $\boldsymbol{u}_3 \in \mathbb{R}^3$ such that $\|\boldsymbol{u}_3\| = 1$ and $\langle \boldsymbol{u}_1, \boldsymbol{u}_3 \rangle = \langle \boldsymbol{u}_2, \boldsymbol{u}_3 \rangle = 0$. Determine the eigenvalues and eigenvectors of the matrix

$$A = \boldsymbol{u}_1 \boldsymbol{u}_1^T + \boldsymbol{u}_2 \boldsymbol{u}_2^T$$

[Answer]

2. Suppose A is a $m \times n$ matrix with $m > n$ such that $\det(A^T A) \neq 0$. Determine the algebraic multiplicity of the eigenvalue $\lambda = 0$ for AA^T.

[Answer]

3. **True** or **False**: Let $\lambda_1, \lambda_2 \in \mathbb{R}$ such that $\lambda_1 \neq \lambda_2$, and let $\boldsymbol{v}_1, \boldsymbol{v}_2 \in \mathbb{R}^2$ be nonzero vectors such that $\langle \boldsymbol{v}_1, \boldsymbol{v}_2 \rangle = 0$. There is a <u>unique</u> 2×2 symmetric matrix A such that $A\boldsymbol{v}_1 = \lambda_1$ and $A\boldsymbol{v}_2 = \lambda_2$.

[Answer]

4. **True** or **False**: There exists a 3×3 symmetric matrix A such that the vectors

$$\boldsymbol{v}_1 = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix} \qquad \boldsymbol{v}_2 = \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix} \qquad \boldsymbol{v}_3 = \begin{bmatrix} -4 \\ -1 \\ 7 \end{bmatrix}$$

are eigenvectors of A.

[Answer]

5. **True** or **False**: Let A and B be $n \times n$ symmetric matrices. The matrices AB and BA have the same set of eigenvalues.

[Answer]

6. Let A be a $m \times n$ matrix with $\operatorname{rank}(A) = n$, and let $A = Q_1 R_1$ be the thin QR decomposition. Determine the characteristic polynomial of $Q_1 Q_1^T$.

[Answer]

7. **True** or **False**: Suppose A and B are $n \times n$ matrices with the same characteristic polynomials $\det(A - xI) = \det(B - xI)$. Then $A = B$.

[Answer]

8. **True** or **False**: If λ is an eigenvalue of A, then $\lambda - \sigma$ is an eigenvalue of $A - \sigma I$ for any $\sigma \in \mathbb{R}$.

[Answer]

9. **True** or **False**: If A is a $n \times n$ matrix with characteristic polynomial $c_A(x) = x^{n-m}(x-1)^m$ for some integer $0 < m < n$ then A is a projection matrix onto a subspace $U \subset \mathbb{R}^n$ with dimension m.

[Answer]

10. **True** or **False**: Let A, B be $n \times n$ matrices. If $\boldsymbol{v}$ is an eigenvector for both A, B, then $\boldsymbol{v}$ is an eigenvector for AB.

[Answer]

11. **True** or **False**: Let A be an $n \times n$ matrix. If λ is an eigenvalue of A, then $\lambda + 1$ is an eigenvalue of $A + I$.

[Answer]

12. **True** or **False**: Let A be a $n \times n$ matrix such that $A^T = -A$. Then all eigenvalues of A are purely imaginary (that is, every eigenvalue takes the form $\lambda = bi$ where $b \in \mathbb{R}$ and $i = \sqrt{-1}$).

[Answer]

Diagonalization

1. Suppose A is a 4×4 matrix with characteristic polynomial

$$c_A(x) = (x^2 + x - 2)(x^2 - x - 2)$$

Is A diagonalizable? Justify your answer.

[Answer]

2. Consider the matrix

$$A = \begin{bmatrix} 1 & 5 \\ 5 & 1 \end{bmatrix}$$

(a) Find matrices P and D such that $A = PDP^{-1}$.

(b) Compute the limit

$$\lim_{k\to\infty} \lambda_1^{-k} A^k$$

where λ_1 is the largest eigenvalue (in absolute value). In other words, if λ_1 and λ_2 are the eigenvalues of A then $|\lambda_1| > |\lambda_2|$.

[Answer]

3. Suppose A is a 4×4 matrix with characteristic polynomial

$$c_A(x) = (x^2 + 3x + 2)(x^2 - 3x - 2)$$

Is A diagonalizable? Justify your answer.

[Answer]

4. Consider the matrix

$$A = \begin{bmatrix} 2 & 6 \\ 6 & 2 \end{bmatrix}.$$

(a) Find matrices P and D such that $A = PDP^{-1}$.

(b) Compute the limit

$$\lim_{k\to\infty} \lambda_1^{-k} A^k$$

where λ_1 is the largest eigenvalue (in absolute value). In other words, if λ_1 and λ_2 are the eigenvalues of A then $|\lambda_1| > |\lambda_2|$.

[Answer]

5. **True** or **False**: If A is a 3×3 matrix with eigenvalues $\lambda_1, \lambda_2, \lambda_3 \in \mathbb{R}$ and corresponding eigenvectors

$$\boldsymbol{v}_1 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \qquad \boldsymbol{v}_2 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \qquad \boldsymbol{v}_3 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

then A is a symmetric matrix.

[Answer]

6. Consider the matrix

$$A = \begin{bmatrix} 2 & -2 \\ -2 & 5 \end{bmatrix}.$$

(a) Find matrices P and D such that $A = PDP^{-1}$.

(b) Compute the limit

$$\lim_{k \to \infty} \lambda_1^{-k} A^k$$

where λ_1 is the largest eigenvalue (in absolute value). In other words, if λ_1 and λ_2 are the eigenvalues of A then $|\lambda_1| > |\lambda_2|$.

[Answer]

7. **True** or **False**: If A is a 3×3 matrix with eigenvalues $\lambda_1, \lambda_2, \lambda_3 \in \mathbb{R}$ and corresponding eigenvectors

$$\boldsymbol{v}_1 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \qquad \boldsymbol{v}_2 = \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix} \qquad \boldsymbol{v}_3 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}.$$

then A is a symmetric matrix.

[Answer]

8. Consider the matrix

$$A = \begin{bmatrix} 3 & -2 \\ -2 & 0 \end{bmatrix}.$$

(a) Find matrices P and D such that $A = PDP^{-1}$.

(b) Compute the limit

$$\lim_{k \to \infty} \lambda_1^{-k} A^k$$

where λ_1 is the largest eigenvalue (in absolute value). In other words, if λ_1 and λ_2 are the eigenvalues of A then $|\lambda_1| > |\lambda_2|$.

[Answer]

9. Determine all values c such that the matrix

$$A = \begin{bmatrix} 3 & c \\ -1 & 5 \end{bmatrix}$$

is orthogonally diagonalizable. In other words, find all possible values c such that there exists a diagonal matrix D and orthogonal matrix P such that $A = PDP^T$.

[Answer]

10. **True** or **False**: If A is any 4×4 matrix with characteristic polynomial

$$c_A(x) = (x^2 + 2x - 2)(x^2 - 2x - 2)$$

then A is diagonalizable.

[Answer]

11. Find the orthogonal diagonalization $A = PDP^T$ of the matrix

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 3 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

The characteristic polynomial of A is $c_A(x) = x^3 - 5x^2 + 4x$.

[Answer]

12. **True** or **False**: There exists a diagonalizable matrix A with characteristic polynomial

$$c_A(x) = (x-1)^2(x-2)^2$$

[Answer]

13. **True** or **False**: There exists a 3×3 symmetric matrix A with distinct eigenvalues such that the vectors

$$\boldsymbol{v}_1 = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix} \qquad \boldsymbol{v}_2 = \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix} \qquad \boldsymbol{v}_3 = \begin{bmatrix} 1 \\ 0 \\ -3 \end{bmatrix}$$

are eigenvectors of A.

[Answer]

14. **True** or **False**: If A and B are $n \times n$ diagonalizable matrices such that any eigenvector of A is also an eigenvector of B, and any eigenvector of B is also an eigenvector of A, then $AB = BA$.

[Answer]

15. **True** or **False**: If A is a diagonalizable matrix, then A is invertible.

[Answer]

16. Find the unique symmetric matrix A with eigenvalues $\lambda_1 = 1$, $\lambda_2 = 3$, $\lambda_3 = -1$ and eigenvectors $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$ corresponding to λ_1 and λ_2 respectively where

$$\boldsymbol{v}_1 = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} \qquad \boldsymbol{v}_2 = \begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix}$$

[Answer]

17. Consider the matrix

$$A = \begin{bmatrix} 4 & 1 & 1 \\ -1 & 2 & 1 \\ 0 & 0 & 3 \end{bmatrix}$$

with characteristic polynomial $c_A(x) = (x-3)^3$. Is A diagonalizable? Explain.

[Answer]

18. **True** or **False**: Let P be the projection matrix onto a subspace $U \subset \mathbb{R}^n$. Then P is diagonalizable.

[Answer]

19. The matrix

$$A = \begin{bmatrix} 4 & 2 & -2 \\ 2 & -1 & 1 \\ -2 & 1 & 7 \end{bmatrix}$$

has distinct eigenvalues λ_1, λ_2 and λ_3 where $\lambda_1 = 4$ and $\lambda_2 = -2$ with corresponding eigenvectors

$$\boldsymbol{v}_1 = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} \qquad \boldsymbol{v}_2 = \begin{bmatrix} 2 \\ -5 \\ 1 \end{bmatrix}$$

Find the eigenvalue λ_3.

[Answer]

20. **True** or **False**: If A is a $n \times n$ symmetric matrix such that $|\lambda_i| < 1$ for all eigenvalues $\lambda_1, \dots, \lambda_n$ then $A^k \to \mathbf{0}$ as $k \to \infty$ (where $\mathbf{0}$ is the zero matrix of size n).

[Answer]

21. **True** or **False**: Let $\boldsymbol{v}_1, \dots, \boldsymbol{v}_n \in \mathbb{R}^n$ be an orthogonal set of (nonzero) vectors. There is a unique $n \times n$ symmetric matrix A such that $\boldsymbol{v}_1, \dots, \boldsymbol{v}_n$ are eigenvectors of A.

[Answer]

22. Let A be a 2×2 matrix. Suppose that

$$N(A - I) = \text{span}\left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix} \right\} \qquad N(A + 2I) = \text{span}\left\{ \begin{bmatrix} 3 \\ 3 \end{bmatrix} \right\}$$

Find an invertible 2×2 matrix S and a diagonal matrix D such that $A = SDS^{-1}$.

[Answer]

23. **True** or **False**: Consider the matrix

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & -2024 & -2024 & 3 \\ 3 & -2024 & -2024 & 2 \\ 4 & 3 & 2 & 1 \end{bmatrix}$$

All eigenvalues of A are real and A is diagonalizable.

[Answer]

24. Let $A = SDS^{-1}$ where

$$S = \begin{bmatrix} 1 & 4 \\ 2 & 4 \end{bmatrix} \qquad D = \begin{bmatrix} -2 & 0 \\ 0 & 4 \end{bmatrix}$$

Find the eigenvalues of A^2 and a basis for $N(A^2 - 4I)$.

[Answer]

25. **True** or **False**: Consider the matrix

$$A = \begin{bmatrix} 2 & 0 & 2 & 4 \\ 0 & 2 & 0 & 2 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}$$

Then A is diagonalizable.

[Answer]

26. Let A be a 3×3 symmetric matrix with eigenvalues $\lambda_1 = -3$, $\lambda_2 = -1$, $\lambda_3 = 5$ and corresponding eigenvectors $\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3$ where

$$\boldsymbol{v}_1 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \qquad \boldsymbol{v}_2 = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$$

(a) Find $\boldsymbol{v}_3$.

(b) Determine P and D in the orthogonal diagonalization PDP^T of A.

(c) Determine the eigenvectors, and corresponding eigenvalues, for A^{-1}.

[Answer]

27. Let $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^n$ such that $\langle \boldsymbol{x}, \boldsymbol{y} \rangle = 0$ and $\|\boldsymbol{x}\| = \|\boldsymbol{y}\| = 3$. Let $A = \boldsymbol{x}\boldsymbol{y}^T$.

(a) Compute $\|\boldsymbol{x} + 2\boldsymbol{y}\|$.

(b) Determine $\operatorname{rank}(A)$.

(c) Determine $\|A\|$.

(d) Is A diagonalizable? Justify your answer.

[Answer]

28. Calculate $A^{-2} + 2A^5$ for the matrix

$$A = U \begin{bmatrix} 1 & 0 \\ 0 & -3 \end{bmatrix} U^T$$

where U is an orthogonal matrix. (Note: your answer should depend on U.)

[Answer]

Singular Value Decomposition

1. **True** or **False**: Suppose $A = P\Sigma Q^T$ is the singular value decomposition of A such that Q is a permutation matrix. Then the columns of A are orthogonal. (Recall that a permutation matrix is any matrix obtained from the identity matrix by permuting the rows.)

[Answer]

2. **True** or **False**: Suppose $A = P\Sigma Q^T$ is the singular value decomposition of A such that Q is a permutation matrix. Then the rows of A are orthogonal. (Recall that a permutation matrix is any matrix obtained from the identity matrix by permuting the rows.)

[Answer]

3. Consider the matrix

$$A = \frac{1}{2} \begin{bmatrix} 1 & 2 & 3 & 1 \\ -1 & 2 & -1 & -3 \\ -1 & 2 & 1 & 3 \\ 1 & 2 & -3 & -1 \end{bmatrix}$$

   (a) (5 marks) Compute the condition number of A. (Hint: consider A^TA not AA^T.)

   (b) (5 marks) Find a *unit* vector $\boldsymbol{x}$ such that $\|A\boldsymbol{x}\| = \|A\|$.

[Answer]

4. Find the singular values of the matrix

$$A = \begin{bmatrix} 2 & 0 & 1 \\ -1 & 1 & 1 \\ 1 & -1 & -1 \\ -2 & -1 & 0 \end{bmatrix}$$

[Answer]

5. Let A be a 4×4 matrix with singular value decomposition $A = P\Sigma Q^T$ where

$$P = \begin{bmatrix} \boldsymbol{p}_1 & \boldsymbol{p}_2 & \boldsymbol{p}_3 & \boldsymbol{p}_4 \end{bmatrix} \qquad \Sigma = \begin{bmatrix} 5 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad Q = \begin{bmatrix} \boldsymbol{q}_1 & \boldsymbol{q}_2 & \boldsymbol{q}_3 & \boldsymbol{q}_4 \end{bmatrix}$$

Let $\boldsymbol{x} = \boldsymbol{q}_1 + \boldsymbol{q}_2 + \boldsymbol{q}_3 + \boldsymbol{q}_4$. Compute $\|A\boldsymbol{x}\|$.

[Answer]

6. Compute the singular value decomposition of the matrix

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & -1 \\ 1 & 1 & -1 \end{bmatrix}$$

Note: the characteristic polynomial of both AA^T and A^TA is $(x-1)(x-4)^2$.

[Answer]

7. Suppose A is 4×4 matrix such that A^TA has eigenvalues

$$\lambda_1 = 10 \ , \ \lambda_2 = 4 \ , \ \lambda_3 = 3 \ , \ \lambda_4 = 1$$

with corresponding (unit) eigenvectors $\boldsymbol{q}_1$, $\boldsymbol{q}_2$, $\boldsymbol{q}_3$, $\boldsymbol{q}_4$. Let $\boldsymbol{v} = 2\boldsymbol{q}_3 - 5\boldsymbol{q}_1$. Determine $\|A\boldsymbol{v}\|$.

[Answer]

8. Suppose A is 4×4 matrix such that A^TA has eigenvalues

$$\lambda_1 = 10 \ , \ \lambda_2 = 4 \ , \ \lambda_3 = 3 \ , \ \lambda_4 = 1$$

with corresponding (unit) eigenvectors $\boldsymbol{q}_1$, $\boldsymbol{q}_2$, $\boldsymbol{q}_3$, $\boldsymbol{q}_4$. Let $\boldsymbol{v} = 2\boldsymbol{q}_1 + \boldsymbol{q}_4$. Determine $\|A\boldsymbol{v}\|$.

[Answer]

9. Suppose A is a 5×5 symmetric matrix with eigenvalues $5, 2, 0, -1, -6$. Determine $\|A\|$.

[Answer]

10. Find all possible values c such that $\|A\| = 3$ where

$$A = \begin{bmatrix} 1 & -c \\ c & 0 \end{bmatrix}$$

[Answer]

11. **True** or **False**: If A is any matrix then $\|A\| = \|A^T\|$.

[Answer]

12. Let A be a matrix with the singular value decomposition $A = P\Sigma Q^T$ where

$$P = \frac{1}{3}\begin{bmatrix} 1 & 2 & -2 \\ 2 & 1 & 2 \\ -2 & 2 & 1 \end{bmatrix} \quad \Sigma = \begin{bmatrix} 3 & 0 & 0 & 0 \\ 0 & \sqrt{2} & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad Q = \begin{bmatrix} 1/2 & 0 & -5/6 & \sqrt{2}/6 \\ 1/2 & \sqrt{2}/2 & 1/6 & -\sqrt{2}/3 \\ 1/2 & -\sqrt{2}/2 & 1/6 & -\sqrt{2}/3 \\ 1/2 & 0 & 1/2 & \sqrt{2}/2 \end{bmatrix}$$

(a) Determine $\|A\|$.

(b) Find a basis of $R(A)$.

(c) Find a basis of $N(A)$.

[Answer]

13. **True** or **False**: Suppose $A = P\Sigma_1 Q^T$ and $B = P\Sigma_2 Q^T$ are the singular value decompositions of $m \times n$ matrices A and B. In particular, both matrices P and Q are the same in both decompositions but the singular values may be different $\Sigma_1 \neq \Sigma_2$. Then $\|A + B\| = \|A\| + \|B\|$.

[Answer]

14. Find a unit vector $\boldsymbol{x}$ such that $\|A\boldsymbol{x}\|$ is as small as possible for the matrix

$$A = \begin{bmatrix} 2 & 0 & -1 \\ 0 & 2 & 1 \\ -1 & 1 & 2 \end{bmatrix}$$

The characteristic polynomial of A is $c_A(x) = (2 - x)(x^2 - 4x + 2)$.

[Answer]

15. Compute $\|A\|$ for the matrix

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & -1 \\ 1 & 2 & 0 \\ 0 & 1 & -1 \end{bmatrix}$$

[Answer]

16. **True** or **False**: Let A be an invertible matrix. If λ is an eigenvalue of $A^T A$ then λ is a real number and $\lambda > 0$.

[Answer]

17. **True** or **False**: If $A = P\Sigma Q^T$ is the singular value decomposition A then $\|A\| = \|\Sigma\|$.

[Answer]

18. Suppose $A = P\Sigma_1 Q^T$ and $B = P\Sigma_2 Q^T$ are the singular value decompositions of A and B such that the orthogonal matrices P and Q are the same in both decompositions however Σ_1 and Σ_2 are given by

$$\Sigma_1 = \begin{bmatrix} 10 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 5 \end{bmatrix} \qquad \Sigma_2 = \begin{bmatrix} 7 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Determine $\|A - B\|$.

[Answer]

19. Consider the matrix $A = \begin{bmatrix} 1 & 2 \\ 1 & -2 \end{bmatrix}$.

    (a) Find the singular value decomposition of A.

    (b) Sketch the set of points $\{A\boldsymbol{x} : \|\boldsymbol{x}\| = 1\}$.

[Answer]

20. Compute $\text{cond}(A)$ for the matrix

$$A = \begin{bmatrix} 1 & 1 & -1 \\ 1 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix}$$

[Answer]

21. Let

$$A = \begin{bmatrix} 0 & -6 & 0 \\ 0 & 0 & 3 \\ 2 & 0 & 0 \end{bmatrix} \qquad \boldsymbol{x} = \begin{bmatrix} -2 \\ 2 \\ 1 \end{bmatrix}$$

    (a) Compute the condition number of A.

    (b) Compute $\|A\boldsymbol{x}\|_2$ and $\|A\boldsymbol{x}\|_1$.

    (c) Give an example of a 2×2 matrix B such that $B^2 = I$ and $\text{cond}(B) \neq 1$.

[Answer]

22. Let $A = \begin{bmatrix} 3 & -1 \\ 3 & 1 \end{bmatrix}$.

    (a) Compute the singular value decomposition of A.

    (b) Find $\|A\|$.

[Answer]

23. Let A have the singular value decomposition $A = P\Sigma Q^T$ where

$$P = \frac{1}{2}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix} \qquad \Sigma = \begin{bmatrix} 3 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad Q = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

(a) Find $\|P\|$.

(b) Find the dimension of the null space of A.

(c) Compute $\|A^TAA^TA\|$.

[Answer]

24. **True** or **False**: The following matrix is diagonalizable

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 5 & 6 & 7 \\ 0 & 0 & 8 & 9 \\ 0 & 0 & 0 & 10 \end{bmatrix}$$

[Answer]

25. Let $A = \begin{bmatrix} 2 & -2 \\ 1 & 1 \end{bmatrix}$.

(a) Compute the singular value decomposition of A.

(b) Find $\|A\|$.

[Answer]

26. **True** or **False**: Let A be an $n \times n$ matrix. If all eigenvalues of A are positive real numbers, then $\|A\|$ equals its largest eigenvalue.

[Answer]

27. Give an example of 4×4 matrix A such that $R(A) = N(A)$, and compute its SVD expansion.

[Answer]

28. Let A be a 3×3 matrix and suppose $A^TA = QDQ^T$ where Q is orthogonal and

$$D = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 4 \end{bmatrix}$$

(a) Find the singular values of A.

(b) Determine the condition number of $A^T A + (A^T A)^2 + I$ (where I is the 3 by 3 identity matrix).

(c) Find the singular values of $(A^T A A^T)^{-1}$.

[Answer]

29. Compute the singular value decomposition of the matrix

$$A = \begin{bmatrix} 1 & 2 \\ -2 & 1 \\ 3 & 1 \end{bmatrix}$$

[Answer]

30. **True** or **False**: If A is a real symmetric matrix, then its orthogonal diagonalization is identical to its SVD.

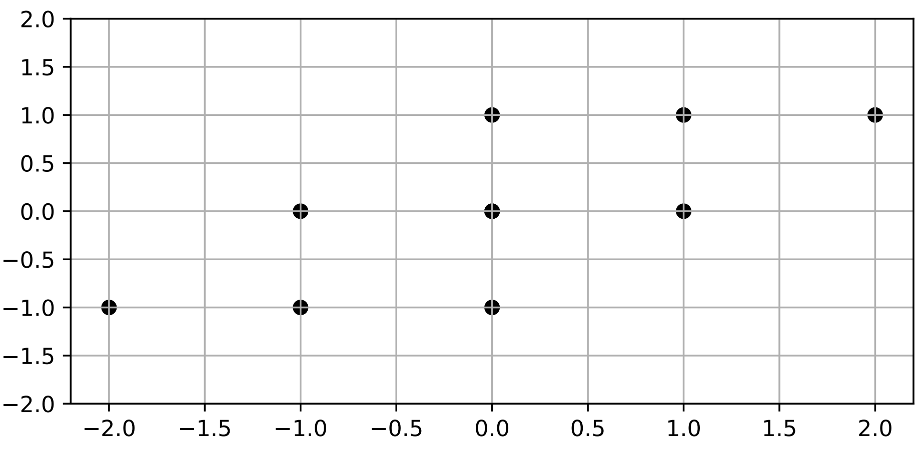
[Answer]

Principal Component Analysis

1. **True** or **False**: Let X be a (normalized) data matrix, let $\boldsymbol{x}$ be a row of X, let $\boldsymbol{w}_1$ be the first weight vector of X and let $\boldsymbol{w}_2$ be the second weight vector of X. If $\langle \boldsymbol{x}, \boldsymbol{w}_1 \rangle \neq 0$ then $|\langle \boldsymbol{x}, \boldsymbol{w}_2 \rangle| < \|\boldsymbol{x}\|$.

[Answer]

2. Find the weight vectors $\boldsymbol{w}_1$ and $\boldsymbol{w}_2$ for the dataset displayed below



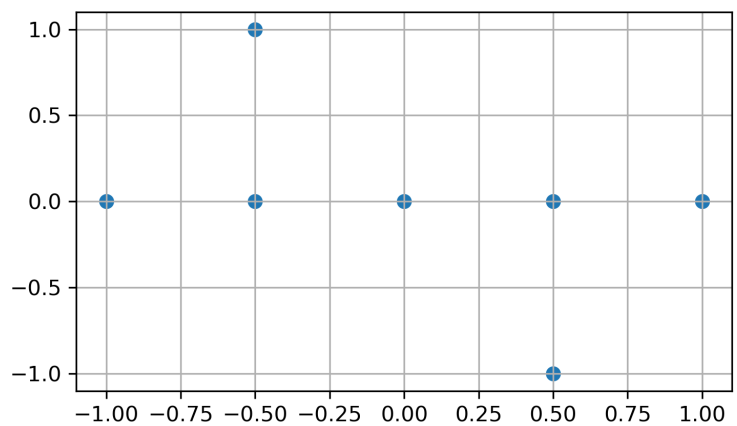
[Answer]

3. Find the weight vectors $\boldsymbol{w}_1$ and $\boldsymbol{w}_2$ for the data points:

$$(-2, 0)\ ,\ (-1, 1)\ ,\ (0, 1)\ ,\ (1, -1)\ ,\ (2, -1)$$

[Answer]

4. Find the first weight vector $\boldsymbol{w}_1$ for the dataset displayed below:



[Answer]

5. **True** or **False**: Let X be a $n \times p$ (normalized) data matrix and let $\boldsymbol{x}_i$ and $\boldsymbol{x}_j$ be two different rows of X such that $\langle \boldsymbol{x}_i, \boldsymbol{x}_j \rangle = 0$. If $\boldsymbol{w}_1$ is the first weight vector of X and $\langle \boldsymbol{x}_i, \boldsymbol{w}_1 \rangle = 0$ then $\langle \boldsymbol{x}_j, \boldsymbol{w}_1 \rangle = 0$.

[Answer]

6. **True** or **False**: Let X be a $n \times p$ (normalized) data matrix and let $\boldsymbol{x}_i$ and $\boldsymbol{x}_j$ be two different rows of X such that $\|\boldsymbol{x}_i\| < \|\boldsymbol{x}_j\|$. If $\boldsymbol{w}_1$ is the first weight vector of X, then $|\langle \boldsymbol{x}_i, \boldsymbol{w}_1 \rangle| < |\langle \boldsymbol{x}_j, \boldsymbol{w}_1 \rangle|$.

[Answer]

7. **True** or **False**: Let X be a $n \times p$ (normalized) data matrix, let $\boldsymbol{x}_i$ be a row of X and let $\boldsymbol{w}_1$ be the first weight vector. If $\|\boldsymbol{x}_i\| > 0$ then $|\langle \boldsymbol{x}_i, \boldsymbol{w}_1 \rangle| > 0$.

[Answer]

8. **True** or **False**: Let X be a $n \times p$ (normalized) data matrix, let $\boldsymbol{x}_i$ be a row of X and let $\boldsymbol{w}_1$ be the first weight vector. Then $|\langle \boldsymbol{x}_i, \boldsymbol{w}_1 \rangle| \leq \|\boldsymbol{x}_i\|$.

[Answer]

9. **True** or **False**: Let X be a $n \times p$ (normalized) data matrix with $p > 2$, let $\boldsymbol{x}_i$ be a row of X, let $\boldsymbol{w}_1$ be the first weight vector, and let $\boldsymbol{w}_2$ be the second weight vector. Then $|\langle \boldsymbol{x}_i, \boldsymbol{w}_2 \rangle| \leq |\langle \boldsymbol{x}_i, \boldsymbol{w}_1 \rangle|$.

[Answer]

10. **True** or **False**: Let X be a $n \times p$ (normalized) data matrix, let $\boldsymbol{x}_i$ be a row of X and let $\boldsymbol{w}_1$ be the first weight vector. Then $\langle \boldsymbol{x}_i, \boldsymbol{w}_1 \rangle \geq 0$.

[Answer]

Power Method

1. Let A be a 3×3 matrix such that A is *not* a diagonal matrix and the eigenvalues of A are $\lambda_1 = 1$, $\lambda_2 = 1/2$ and $\lambda_3 = -6$. Let $\boldsymbol{x}_0 \in \mathbb{R}^3$ be a random nonzero vector and let $\boldsymbol{x}_k = A^{-k}\boldsymbol{x}_0$. Determine the (most likely) value c such that

$$\frac{\langle \boldsymbol{x}_k, \boldsymbol{x}_{k+1} \rangle}{\langle \boldsymbol{x}_k, \boldsymbol{x}_k \rangle} \to c \quad \text{as} \quad k \to \infty$$

[Answer]

2. Use at least 3 iterations of the power method to approximate the dominant eigenvalue and corresponding eigenvector of the matrix

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

[Answer]

Discrete Fourier Transform

Complex Numbers, Vectors and Matrices

1. **True** or **False**: If A is a complex hermitian matrix then the diagonal entries of A are real numbers.

[Answer]

2. **True** or **False**: Let $\boldsymbol{v}_1, \dots, \boldsymbol{v}_m \in \mathbb{C}^N$ be complex orthogonal vectors and let A be the matrix with $\boldsymbol{v}_1, \dots, \boldsymbol{v}_m$ in the columns

$$A = \begin{bmatrix} \boldsymbol{v}_1 & \cdots & \boldsymbol{v}_m \end{bmatrix}$$

Then $A^T \bar{A}$ is a diagonal matrix.

[Answer]

3. Does there exist a 2×2 complex unitary matrix U of the form

$$U = \begin{bmatrix} x & y \\ w & z \end{bmatrix}$$

such that $x^2 + y^2 = 0$? If yes, find an example. If no, prove why no such matrix exists.

[Answer]

4. Find all $\theta \in \mathbb{R}$ (if any) that solve the equation

$$e^{i\pi\theta} = \frac{\pi}{2} + i\frac{\pi}{2}$$

[Answer]

5. Compute $\|\boldsymbol{u}\|$ for the vector

$$\boldsymbol{u} = \begin{bmatrix} 1+i \\ 1-i \\ 3 \end{bmatrix}$$

[Answer]

Discrete Fourier Transform

1. **True** or **False**: If $\text{DFT}(\boldsymbol{x}) = \text{DFT}(\boldsymbol{y})$ then $\boldsymbol{x} = \boldsymbol{y}$.

[Answer]

2. **True** or **False**: Let $\boldsymbol{x}_1, \boldsymbol{x}_2 \in \mathbb{R}^N$ and let $\boldsymbol{y}_1 = \text{DFT}(\boldsymbol{x}_1)$ and $\boldsymbol{y}_2 = \text{DFT}(\boldsymbol{x}_2)$. If $\langle \boldsymbol{x}_1, \boldsymbol{x}_2 \rangle = 0$ then $\langle \boldsymbol{y}_1, \boldsymbol{y}_2 \rangle = 0$.

[Answer]

3. **True** or **False**: Let $\boldsymbol{x}_1, \boldsymbol{x}_2 \in \mathbb{R}^N$ and let $\boldsymbol{y}_1 = \text{DFT}(\boldsymbol{x}_1)$ and $\boldsymbol{y}_2 = \text{DFT}(\boldsymbol{x}_2)$. Then $\langle \boldsymbol{x}_1, \boldsymbol{x}_2 \rangle = \langle \boldsymbol{y}_1, \boldsymbol{y}_2 \rangle$.

[Answer]

4. **True** or **False**: Let $\boldsymbol{x} \in \mathbb{R}^N$ such that $\boldsymbol{x}[k] > 0$ for all $k = 0, \ldots, N-1$. Let $\boldsymbol{y} = \text{DFT}(\boldsymbol{x})$. Then $\boldsymbol{y}[0] > 0$.

[Answer]

5. Determine the inner product $\langle \boldsymbol{x}_1, \boldsymbol{x}_2 \rangle$ for vectors $\boldsymbol{x}_1, \boldsymbol{x}_2 \in \mathbb{R}^8$ given the discrete Fourier transforms

$$\text{DFT}(\boldsymbol{x}_1) = \begin{bmatrix} 1 \\ 1+i \\ 0 \\ 2-i \\ -1 \\ 2+i \\ 0 \\ 1-i \end{bmatrix} \qquad \text{DFT}(\boldsymbol{x}_1) = \begin{bmatrix} 0 \\ 1-2i \\ -3-i \\ 0 \\ 1 \\ 0 \\ -3+i \\ 1+2i \end{bmatrix}$$

Hint: $\frac{1}{N}\overline{F}_N^T F_N = I$.

[Answer]

Sinusoids

1. Compute $\text{DFT}(\boldsymbol{x})$ where $\boldsymbol{x} = 5\cos(4\pi \boldsymbol{t} + \pi/2) \in \mathbb{C}^8$. Recall, the vector $\boldsymbol{t}$ is

$$\boldsymbol{t} = \begin{bmatrix} 0 & 1/8 & 1/4 & 3/8 & 1/2 & 5/8 & 3/4 & 7/8 \end{bmatrix}^T$$

[Answer]

2. Let $\boldsymbol{y} \in \mathbb{C}^8$ such that

$$\boldsymbol{y} = \begin{bmatrix} 1 \\ 0 \\ 2-2i \\ 1+i\sqrt{3} \\ 0 \\ 1-i\sqrt{3} \\ 2+2i \\ 0 \end{bmatrix}$$

Find values A_0, A_1, A_2, k_1, k_2, ϕ_1, ϕ_2 such that $\boldsymbol{y} = \text{DFT}(\boldsymbol{x})$ where $\boldsymbol{x}$ is of the form

$$\boldsymbol{x} = A_0 + A_1 \cos\left(2\pi k_1 \boldsymbol{t} + \phi_1\right) + A_2 \cos\left(2\pi k_2 \boldsymbol{t} + \phi_2\right) \quad , \quad k_1 < k_2$$

Recall $\boldsymbol{t} \in \mathbb{C}^N$ is the vector

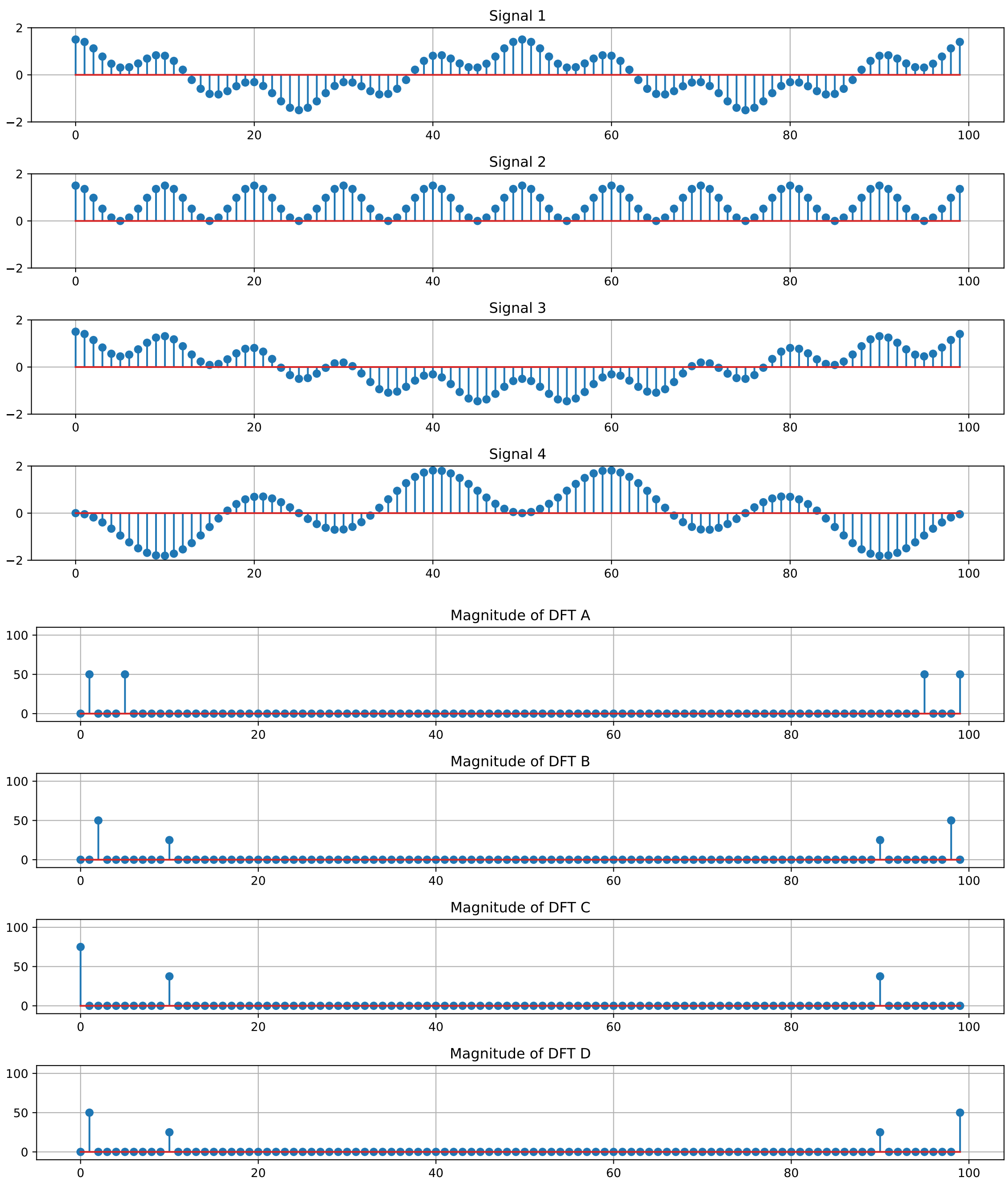
$$\boldsymbol{t} = \begin{bmatrix} 0 \\ 1/N \\ 2/N \\ \vdots \\ (N-1)/N \end{bmatrix}$$

[Answer]

3. Let $\boldsymbol{x} \in \mathbb{R}^{16}$ such that $\boldsymbol{y} = \text{DFT}(\boldsymbol{x})$ where $\boldsymbol{y}[0] = 8$, $\boldsymbol{y}[4] = \boldsymbol{y}[12] = 4$, and all other entries of $\boldsymbol{y}$ are zero. Sketch the stemplot of $\boldsymbol{x}$.

[Answer]

4. Match the signal with its discrete Fourier transform.

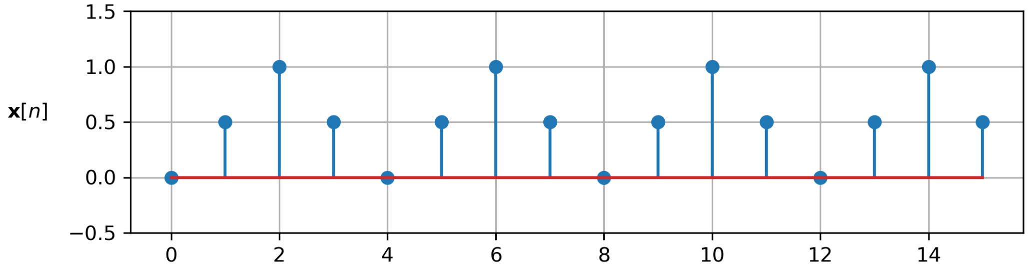


Signal 1 = DFT ________ Signal 2 = DFT ________

Signal 3 = DFT ________ Signal 4 = DFT ________

[Answer]

5. Compute $\boldsymbol{y} = \mathrm{DFT}(\boldsymbol{x})$ for the signal $\boldsymbol{x} \in \mathbb{R}^{16}$ displayed in the figure below:



Hint: write $\boldsymbol{x}$ as a sum of sinusoids.

[Answer]

6. Give an example of a vector $\boldsymbol{x} \in \mathbb{R}^{16}$ which satisfies both conditions:

   - $\boldsymbol{x}[k] = 1$ or $\boldsymbol{x}[k] = -1$ for each $k = 0, \ldots, 15$
   - $\boldsymbol{y}[0] = \boldsymbol{y}[8] = 0$ where $\boldsymbol{y} = \text{DFT}(\boldsymbol{x})$

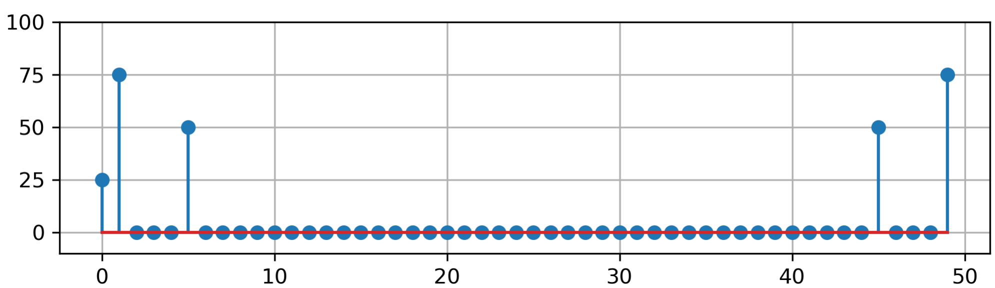
   You may write out the entries of $\boldsymbol{x}$ or sketch a stemplot.

[Answer]

7. **True** or **False**: $\langle \cos(2\pi k\boldsymbol{t}), \cos(2\pi \ell \boldsymbol{t}) \rangle = 0$ for any integers $k \neq \ell$.

[Answer]

8. The figure below is a stem plot for a real vector $\boldsymbol{y} \in \mathbb{R}^N$. Write $\boldsymbol{x}$ as a sum of sinusoids where $\boldsymbol{y} = \text{DFT}(\boldsymbol{x})$.



[Answer]

9. Consider the vector

$$\boldsymbol{y} = \begin{bmatrix} 5 \\ 3\sqrt{3} + 3i \\ 0 \\ -2 + 2i \\ 0 \\ -2 - 2i \\ 0 \\ 3\sqrt{3} - 3i \end{bmatrix}$$

(a) Find values $A_0, A_1, A_2, k_1, k_2, \phi_1, \phi_2$ such that $\boldsymbol{y} = \text{DFT}(\boldsymbol{x})$ where

$$\boldsymbol{x} = A_0 + A_1 \cos(2\pi k_1 \boldsymbol{t} + \phi_1) + A_2 \cos(2\pi k_2 \boldsymbol{t} + \phi_2)$$

Recall $\boldsymbol{t} \in \mathbb{C}^N$ is the vector

$$\boldsymbol{t} = \begin{bmatrix} 0 \\ 1/N \\ 2/N \\ \vdots \\ (N-1)/N \end{bmatrix}$$

(b) Plot the magnitude and phase stemplots of $\boldsymbol{y}$.

[Answer]

Answers

Linear Systems

Solving Linear Systems

1. False

[Problem]

2. Unique solution for $k \neq 7$

[Problem]

3. False

[Problem]

4. False

[Problem]

LU Decomposition

1. (a)

$$L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 2 & 0 & 1 & 0 \\ 3 & 0 & -2 & 1 \end{bmatrix} \qquad U = \begin{bmatrix} -3 & 1 & 2 & 0 \\ 0 & 2 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 4 \end{bmatrix}$$

(b) $\det(A) = -24$

[Problem]

2. True

[Problem]

3. (a)

$$L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 4 & 1 & 0 & 0 \\ -2 & 1 & 1 & 0 \\ 1 & 3 & -2 & 1 \end{bmatrix} \qquad U = \begin{bmatrix} 2 & 1 & 1 & 0 \\ 0 & -1 & 4 & 2 \\ 0 & 0 & 3 & -3 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

(b) $\det(A) = 6$

[Problem]

4. (a)

$$L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -4 & 1 & 0 & 0 \\ -3 & 1 & 1 & 0 \\ 2 & -1 & -4 & 1 \end{bmatrix} \qquad U = \begin{bmatrix} 1 & 1 & -2 & 1 \\ 0 & -2 & 3 & -2 \\ 0 & 0 & 2 & -2 \\ 0 & 0 & 0 & 3 \end{bmatrix}$$

(b) $\det(A) = -12$

[Problem]

5. False

[Problem]

6. False

[Problem]

7. True

[Problem]

8. (a)

$$L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ -4 & -3 & 1 & 0 \\ -3 & 3 & -1 & 1 \end{bmatrix} \qquad U = \begin{bmatrix} -4 & 3 & -3 & 1 \\ 0 & 4 & 5 & -1 \\ 0 & 0 & 5 & -1 \\ 0 & 0 & 0 & -4 \end{bmatrix}$$

(b) $\det(A) = 320$

[Problem]

9. True

[Problem]

10. (a)

$$L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 \\ -1 & 0 & \frac{1}{2} & 1 \end{bmatrix} \qquad U = \begin{bmatrix} -3 & 2 & 1 & 0 \\ 0 & 2 & 1 & -1 \\ 0 & 0 & 2 & -4 \\ 0 & 0 & 0 & -3 \end{bmatrix}$$

(b) $\det(A) = 36$

[Problem]

11.

$$\boldsymbol{x} = \begin{bmatrix} 87.5 \\ 60 \\ -37 \\ 50 \end{bmatrix}$$

[Problem]

12.

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -3 & \frac{1}{2} & 1 \end{bmatrix} \qquad U = \begin{bmatrix} 1 & -2 & 2 & 1 \\ 0 & -2 & 2 & 1 \\ 0 & 0 & 0 & -\frac{1}{2} \end{bmatrix}$$

[Problem]

13. False

[Problem]

14. True

[Problem]

15. If $m \leq n$ then $\frac{m(m-1)}{2}$. If $m > n$ then $\frac{n(n-1)}{2} + (m-n)n$.

[Problem]

16. True

[Problem]

17. (a)

$$L = \begin{bmatrix} 1 & 0 \\ c & 1 \end{bmatrix} \qquad U = \begin{bmatrix} 1 & a & b \\ 0 & d - ca & e - cb \end{bmatrix}$$

(b) $d = ca$ and $e = cb$

[Problem]

18. *Under construction*

[Problem]

19. (a) $\det(A) = -6$

(b)

$$\boldsymbol{x} = \begin{bmatrix} -7/2 \\ 3 \\ -1 \end{bmatrix}$$

(c) $-7/2$

[Problem]

Matrix Norm and Condition Number

1. True

[Problem]

2. $||A|| = \max_{\|\boldsymbol{x}\|=1} \|A\boldsymbol{x}\| = 10$

[Problem]

3. True

[Problem]

4. $\text{cond}(A) \approx 1$

[Problem]

5. $\text{cond}(A) = 3$

[Problem]

6. $C = 6$

[Problem]

7. (a) $\|A\| = \sqrt{a^2 + b^2}$

   (b) $\text{cond}(A) = 1$

[Problem]

8. False

[Problem]

9. True

[Problem]

10. False

[Problem]

11. True

[Problem]

12. $$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

[Problem]

13. $\|A\| = 1$ and $\text{cond}(A) = 1$

[Problem]

14. $$A = \begin{bmatrix} 1 & 0 \\ 0 & 1/2 \end{bmatrix}$$

[Problem]

Interpolation

1. A = polynomial, B = cubic spline

[Problem]

2. (a) $b_4 = 0, c_4 = 1, a_4 = -2$

(b) $p''(2.5) = -3$

[Problem]

3. False

[Problem]

4. $\square = 14$

[Problem]

5.

$$A = \begin{bmatrix} 0 & 1 & 0 & 1 \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 1 & 0 \\ 1 & 0 & 0 & -1 \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & -1 & 0 \end{bmatrix} \qquad \boldsymbol{b} = \begin{bmatrix} b+d \\ \frac{a}{\sqrt{2}} + \frac{b}{\sqrt{2}} + c \\ a-d \\ \frac{a}{\sqrt{2}} - \frac{b}{\sqrt{2}} - c \end{bmatrix}$$

[Problem]

6. $p(t) = r(2t^3 + t^2 - 2t + 1), r \in \mathbb{R}$

[Problem]

7. False

[Problem]

8. $5N - 3$ equations

[Problem]

9.

$$p(t) = \frac{13}{18} + \frac{1}{6}t^2 - \frac{1}{9}t^3$$

[Problem]

10.

$$c_3 = 8, \qquad d_3 = 10, \qquad c_4 = -4, \qquad d_4 = 13$$

[Problem]

11.

$$\boldsymbol{c} = \begin{bmatrix} 1 \\ 0 \\ -\frac{a^2+b^2}{a^2b^2} \\ 0 \\ \frac{1}{a^2b^2} \end{bmatrix}$$

[Problem]

12.
$$p(t) = 1 - t + 2t^3$$

[Problem]

13. (a) $x = 7, y = -3, z = -9$
    (b) $p(3.5) = 15/2$

[Problem]

14. *Under construction*

[Problem]

15. Yes, since we have $3N - 1$ equations for $3N$ unknowns.

[Problem]

16. Yes, since the rank of the coefficient matrix is 3.

[Problem]

17.
$$a_4 = -1, \qquad b_4 = 0, \qquad b_5 = -3, \qquad c_6 = -24$$

[Problem]

18.
$$p(t) = \frac{3}{2} + \frac{3}{4}t - \frac{1}{4}t^3$$

[Problem]

19. True

[Problem]

20.
$$f(t) = (\alpha - \gamma) + (\frac{\gamma + \beta}{2})e^t + (\frac{\gamma - \beta}{2})e^{-t}$$

[Problem]

21. *Under construction*

[Problem]

22.
$$p(t) = 2 - 2t + t^3$$

[Problem]

23. False

[Problem]

24.

$$p(t) = 3t^3 - t^2 - 2t + 1$$

[Problem]

25.

$$\int_0^4 p(t)dt = 19$$

[Problem]

26. (a)

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 2 & 4 & 8 & 16 \\ 1 & 3 & 9 & 27 & 81 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} d_0 \\ d_1 \\ d_2 \\ d_3 \\ d_4 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \\ 2 \\ 5 \\ 0 \end{bmatrix}.$$

(b) We know $d_4 = 0$ from the last equation, and so we may effectively remove that equation and d_4. The remaining 4×4 system is invertible since the t-values $0, 1, 2, 3$ are distinct and the corresponding matrix is a Vandermonde matrix.

[Problem]

27. *Under construction*

[Problem]

28. (a) $p(t) = 1 + 2t - t^2$

(b) $\frac{\sqrt{2}}{2} - \frac{3}{4}$

[Problem]

29.

$$a_2 = 0.5, \qquad b_2 = -1.5, \qquad c_1 = 1.5, \qquad c_2 = 0, \qquad d_1 = 1$$

[Problem]

Orthogonality

Subspaces

1. Not a subspace

[Problem]

2. $\text{span}\{\mathbf{u_1}, \mathbf{u_2}\} \neq \text{span}\{\mathbf{u_3}, \mathbf{u_4}\}$

[Problem]

3. $c \neq 7$

[Problem]

4. $c \neq 4$

[Problem]

5. False

[Problem]

6. True

[Problem]

7. False

[Problem]

8. True

[Problem]

9. (a) True
   (b) True
   (c) False
   (d) True

[Problem]

10. (a)

$$L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ -3 & -3 & 1 & 0 \\ 0 & -6 & -1 & 1 \end{bmatrix} \qquad U = \begin{bmatrix} -1 & 1 & 0 & -6 & 2 \\ 0 & -2 & 3 & -2 & 1 \\ 0 & 0 & 0 & 5 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

   (b) Columns $1, 2, 4$ of A (or columns $1, 3, 4$ or $1, 3, 5$, or $1, 2, 5$) or columns $1, 2, 3$ of L

[Problem]

11. True

[Problem]

12. True

[Problem]

13. False

[Problem]

14. *Under construction*

[Problem]

15. True

[Problem]

16. True

[Problem]

17. True

[Problem]

18. (a)

$$L = \begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 4 & -2 & 1 \end{bmatrix}, U = \begin{bmatrix} 1 & 3 & 4 & -1 & -2 \\ 0 & 5 & 3 & 1 & 2 \\ 0 & 0 & 0 & 6 & 5 \end{bmatrix}$$

(b) $\dim(N(A)) = 2$ and $\dim(R(A)) = 3$.

[Problem]

19. Unique solution for $x = 9$, no solution otherwise.

[Problem]

20. False

[Problem]

21. (a)

$$L = \begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 4 & -2 & 1 \end{bmatrix}, U = \begin{bmatrix} 1 & 3 & 4 & -1 & -2 \\ 0 & 5 & 3 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

(b) $\dim(N(A)) = 3$ and $\dim(R(A)) = 2$.

[Problem]

22. True

[Problem]

23. False

[Problem]

24. $\mathrm{span}\{\boldsymbol{u}_1, \boldsymbol{u}_2\} = \mathrm{span}\{\boldsymbol{v}_1, \boldsymbol{v}_2\}$

[Problem]

25. False

[Problem]

26. Not linearly independent, since $\dim(R(A)) = 2$

[Problem]

27. True

[Problem]

28. True

[Problem]

29. True

[Problem]

30. Yes

[Problem]

31. (a)

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & -1/3 & 1 \end{bmatrix}, \qquad U = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & -3 & 3 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

(b)

$$N(A) = \text{span}\{\begin{bmatrix} -1 \\ 1 \\ 1 \\ 0 \end{bmatrix}\}$$

.

[Problem]

32. True

[Problem]

33. $\dim(R(A)) = 2$ and $\dim(N(A)) = n - 2$

[Problem]

34. (a) No
    (b) No
    (c) $\dim(N(A)) = 1$

[Problem]

35. (a) $\{\boldsymbol{x}_1, \boldsymbol{x}_2, \boldsymbol{x}_3, \boldsymbol{x}_4\}$
    (b) $\{\boldsymbol{x}_1, \boldsymbol{x}_2, \boldsymbol{x}_3\}$.
    (c) $\dim N(A) = 0$

[Problem]

Orthogonal Subspaces

1. False

   [Problem]

2. False

   [Problem]

3. *Under construction*

   [Problem]

4. False

   [Problem]

5. False

   [Problem]

6. *Under construction*

   [Problem]

7. *Under construction*

   [Problem]

8. *Under construction*

   [Problem]

Fundamental Subspaces of a Matrix

1. $\dim(N(A)^{\perp}) = 2$

   [Problem]

2. $\dim(R(A)^{\perp}) = 3$

   [Problem]

3. $\dim(N(A)^{\perp}) = 3$

   [Problem]

4. *Under construction*

   [Problem]

5. *Under construction*

   [Problem]

6. True

   [Problem]

7. $\dim(R(A)^{\perp}) = 3$

   [Problem]

8. *Under construction*

[Problem]

9. $\dim(R(A)^{\perp}) = 2$

[Problem]

10.

$$\dim(N(A)) = 1, \qquad \dim(R(A)) = 2, \qquad \dim(N(A^T)) = 4, \qquad \dim(R(A^T)) = 2$$

[Problem]

11. True

[Problem]

Orthogonal Projection

1.

$$P = \begin{bmatrix} 1/2 & 0 & 0 & 1/2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1/2 & 0 & 0 & 1/2 \end{bmatrix}$$

[Problem]

2.

$$P = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1/2 & 1/2 \\ 0 & 0 & 1/2 & 1/2 \end{bmatrix}$$

[Problem]

3. $2\sqrt{14}$

[Problem]

4. $3/\sqrt{11}$

[Problem]

5. True

[Problem]

6. $1/\sqrt{2}$

[Problem]

7. False

[Problem]

8. $\dim(N(P)) = 3$

[Problem]

9.

$$P = \frac{1}{14}\begin{bmatrix} 10 & -6 & 2 & 0 \\ -6 & 5 & 3 & 0 \\ 2 & 3 & 13 & 0 \\ 0 & 0 & 0 & 14 \end{bmatrix}$$

[Problem]

10. False

[Problem]

11. $1/\sqrt{2}$

[Problem]

12. *Under construction*

[Problem]

13.

$$a_1 = 3, \qquad a_2 = 9, \qquad a_3 = 6, \qquad a_4 = 2, \qquad a_5 = 6, \qquad a_6 = 4, \qquad a_7 = 14$$

[Problem]

14. False

[Problem]

15. *Under construction*

[Problem]

16. *Under construction*

[Problem]

17. False

[Problem]

18.

$$\mathrm{proj}_{N(A^T)}(\boldsymbol{x}) = \frac{1}{3}\begin{bmatrix} 1 \\ 0 \\ 1 \\ -1 \end{bmatrix}$$

[Problem]

19.
$$\frac{1}{11}\begin{bmatrix} 2 & 3 & -3 \\ 3 & 10 & 1 \\ -3 & 1 & 10 \end{bmatrix}$$

[Problem]

20. $5/\sqrt{6}$

[Problem]

21. True

[Problem]

22. $\sqrt{2}$

[Problem]

23. False

[Problem]

24.
$$P = \frac{1}{4}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 3 & 1 & -1 \\ 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & 3 \end{bmatrix}$$

[Problem]

25. *Under construction*

[Problem]

26. *Under construction*

[Problem]

27. *Under construction*

[Problem]

28. *Under construction*

[Problem]

29. *Under construction*

[Problem]

30. *Under construction*

[Problem]

QR Decomposition

1. True

[Problem]

2.

$$\text{proj}_{R(A)^\perp}(\boldsymbol{v}) = \frac{5}{4}\begin{bmatrix} -1 \\ 1 \\ -1 \\ 1 \end{bmatrix}$$

[Problem]

3. True

[Problem]

4.

$$\text{proj}_{R(A)^\perp}(\boldsymbol{v}) = -\frac{1}{4}\begin{bmatrix} -1 \\ -1 \\ 1 \\ 1 \end{bmatrix}$$

[Problem]

5.

$$a = -1/\sqrt{2}, \qquad b = 2/3, \qquad c = 1/3$$

[Problem]

6.

$$\text{proj}_{N(A^T)}(\boldsymbol{v}) = \frac{3}{4}\begin{bmatrix} -1 \\ 1 \\ -1 \\ 1 \end{bmatrix}$$

[Problem]

7.

$$\boldsymbol{x} = \begin{bmatrix} 0 \\ -3/2 \\ 3/2 \end{bmatrix}$$

[Problem]

8.

$$a = -2/3, \qquad b = 4/\sqrt{18}, \qquad c = -1/\sqrt{18}$$

[Problem]

9. *Under construction*

[Problem]

10. True

[Problem]

11.

$$Q_1 = \begin{bmatrix} -1/2 & 1/\sqrt{6} & 3\sqrt{44} \\ 1/2 & 2/\sqrt{6} & 1\sqrt{44} \\ 1/2 & 0 & -3\sqrt{44} \\ 1/2 & -1/\sqrt{6} & 5\sqrt{44} \end{bmatrix} \qquad R_1 = \begin{bmatrix} 2 & 0 & -1/2 \\ 0 & \sqrt{6} & 0 \\ 0 & 0 & \sqrt{11}/2 \end{bmatrix}$$

[Problem]

12. $\sqrt{13}$

[Problem]

13. True

[Problem]

14. True

[Problem]

15.

$$a = -1$$

[Problem]

16. True

[Problem]

17. True

[Problem]

18.

$$A = Q_1 R_1 = \begin{bmatrix} 2/3 & 1/\sqrt{3} & -2/3\sqrt{2} \\ 1/3 & 0 & 2/3\sqrt{2} \\ 2/3 & -1/\sqrt{3} & 1/3\sqrt{2} \\ 0 & 1/\sqrt{3} & 1/\sqrt{2} \end{bmatrix} \begin{bmatrix} 3 & 0 & 0 \\ 0 & \sqrt{3} & 0 \\ 0 & 0 & \sqrt{2} \end{bmatrix}$$

[Problem]

19.

$$(A^T A)^{-1} A^T = (R_1^T R_1)^{-1} R_1^T Q_1^T = R_1^{-1} (R_1^T)^{-1} R_1^T Q_1^T = R_1^{-1} Q_1^T$$

[Problem]

20.

$$A = Q_1R_1 = \begin{bmatrix} 1/3 & 0 & 4/\sqrt{22} \\ 0 & -1/2 & 1/\sqrt{22} \\ -2/3 & 1/6 & 0 \\ 0 & -5/6 & -1/\sqrt{22} \\ 2/3 & 1/6 & -2/\sqrt{22} \end{bmatrix} \begin{bmatrix} 3 & 0 & 3 \\ 0 & 6 & -6 \\ 0 & 0 & \sqrt{22} \end{bmatrix}$$

[Problem]

21. True

[Problem]

22. *Under construction*

[Problem]

23. *Under construction*

[Problem]

24. *Under construction*

[Problem]

Least Squares Approximation

1.

$$f(t) = \frac{7}{4} + \frac{1}{2}t$$

[Problem]

2.

$$\boldsymbol{x} = \begin{bmatrix} 7/4 \\ -5/2 \\ 3/2 \end{bmatrix}$$

[Problem]

3. (a)

$$Q_1 = \begin{bmatrix} 1 & -1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \\ -1 & -1 & 1 \end{bmatrix}, \qquad R_1 = \sqrt{3}\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

(b)

$$\boldsymbol{x} = \begin{bmatrix} -1/3 \\ 1/3 \\ 1 \end{bmatrix}$$

[Problem]

4.

$$\boldsymbol{c} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \end{bmatrix}$$

[Problem]

5. (a)

$$\text{proj}_{R(A)^\perp}(\boldsymbol{b}) = \begin{bmatrix} -1/2 \\ 1 \\ -1 \\ 1/2 \end{bmatrix}$$

(b)

$$\boldsymbol{x} = \begin{bmatrix} -13/10 \\ -1/10 \end{bmatrix}$$

[Problem]

6.

$$\boldsymbol{x} = \begin{bmatrix} 2/9 \\ -2/3 \\ -5/3 \end{bmatrix}$$

[Problem]

7.

$$f(t) = \frac{2}{5} + \frac{7}{10}t$$

[Problem]

8.

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1/e & 1/e \\ 1 & 1/e^2 & 2/e^2 \\ 1 & 1/e^3 & 3/e^3 \\ 1 & 1/e^4 & 4/e^4 \\ 1 & 1/e^5 & 5/e^5 \end{bmatrix}, \qquad \boldsymbol{y} = \begin{bmatrix} 0 \\ 3 \\ 2 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

[Problem]

9. (a)

$$\text{proj}_{R(A)^\perp}(\boldsymbol{b}) = \begin{bmatrix} 1/3 \\ 1/3 \\ 0 \\ -1/3 \end{bmatrix}$$

(b)

$$\boldsymbol{x} = \begin{bmatrix} -37/6 \\ 17/6 \\ -1 \end{bmatrix}$$

[Problem]

10. *Under construction*

[Problem]

11. *Under construction*

[Problem]

12. *Under construction*

[Problem]

13. *Under construction*

[Problem]

14. *Under construction*

[Problem]

Eigenvalues and Eigenvectors

Eigenvalues and Eigenvectors

1. Pairs of eigenvalues and corresponding eigenvectors:

$$(a^2, \boldsymbol{u}_1), \qquad (b^2, \boldsymbol{u}_2), \qquad (0, \boldsymbol{u}_3)$$

[Problem]

2. $m - n$

[Problem]

3. True

[Problem]

4. *Under construction*

[Problem]

5. True

[Problem]

6.

$$\pm(x-1)^n x^{m-n}$$

[Problem]

7. False

[Problem]

8. True

[Problem]

9. False

[Problem]

10. *Under construction*

[Problem]

11. *Under construction*

[Problem]

12. *Under construction*

[Problem]

Diagonalization

1. Yes

[Problem]

2. (a)

$$P = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}, \qquad D = \begin{bmatrix} 6 & 0 \\ 0 & -4 \end{bmatrix}$$

(b)

$$\lim_{k \to \infty} \lambda_1^{-k} A^k = \frac{1}{2} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

[Problem]

3. Yes

[Problem]

4. (a)

$$P = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}, \qquad D = \begin{bmatrix} 8 & 0 \\ 0 & -4 \end{bmatrix}$$

(b)

$$\lim_{k \to \infty} \lambda_1^{-k} A^k = \frac{1}{2} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

[Problem]

5. True

[Problem]

6. (a)

$$P = \frac{1}{\sqrt{5}} \begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix}, \qquad D = \begin{bmatrix} 6 & 0 \\ 0 & 1 \end{bmatrix}$$

(b)

$$\lim_{k\to\infty} \lambda_1^{-k} A^k = \frac{1}{5} \begin{bmatrix} 1 & -2 \\ -2 & 4 \end{bmatrix}$$

[Problem]

7. False

[Problem]

8. *Under construction*

[Problem]

9. $c = -1$

[Problem]

10. *Under construction*

[Problem]

11. *Under construction*

[Problem]

12. True

[Problem]

13. False

[Problem]

14. True

[Problem]

15. False

[Problem]

16.

$$A = \frac{1}{3} \begin{bmatrix} 3 & 4 & -4 \\ 4 & 1 & 0 \\ -4 & 0 & 5 \end{bmatrix}$$

[Problem]

17. No (by finding eigenvectors)

[Problem]

18. True

[Problem]

19.
$$\lambda_3 = 8$$

[Problem]

20. True

[Problem]

21. False

[Problem]

22. *Under construction*

[Problem]

23. *Under construction*

[Problem]

24. *Under construction*

[Problem]

25. *Under construction*

[Problem]

26. *Under construction*

[Problem]

27. *Under construction*

[Problem]

28. *Under construction*

[Problem]

Singular Value Decomposition

1. True

[Problem]

2. False

[Problem]

3. (a) $\mathrm{cond}(A) = 2\sqrt{2}$

   (b) $\boldsymbol{x} = \frac{1}{\sqrt{2}} \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}$

[Problem]

4. $$\sigma_1 = \sqrt{10}, \qquad \sigma_2 = \sqrt{5}, \qquad \sigma_3 = 1$$

[Problem]

5. $\|A\boldsymbol{x}\| = \sqrt{30}$

[Problem]

6. $$A = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{6} & 1/\sqrt{3} \\ -1/\sqrt{2} & 1/\sqrt{6} & 1/\sqrt{3} \\ 0 & 2/\sqrt{6} & -1/\sqrt{3} \end{bmatrix} \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 2/\sqrt{6} & 1/\sqrt{3} \\ 1/\sqrt{2} & 1/\sqrt{6} & -1/\sqrt{3} \\ 1/\sqrt{2} & -1/\sqrt{6} & 1/\sqrt{3} \end{bmatrix}^T$$

[Problem]

7. $$\|A\boldsymbol{v}\| = \sqrt{262}$$

[Problem]

8. *Under construction*

[Problem]

9. $$\|A\| = 6$$

[Problem]

10. $$c = \pm\sqrt{6}$$

[Problem]

11. True

[Problem]

12. (a) $\|A\| = 3$

    (b) $$\begin{bmatrix} 1 \\ 2 \\ -2 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}$$

    (c) $$\begin{bmatrix} -5 \\ 1 \\ 1 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ -2 \\ -2 \\ 3 \end{bmatrix}$$

[Problem]

13. True

[Problem]

14.

$$\boldsymbol{x} = \pm \begin{bmatrix} 1/2 \\ -1/2 \\ 1/\sqrt{2} \end{bmatrix}$$

[Problem]

15.

$$\|A\| = \sqrt{\frac{9+3\sqrt{5}}{2}}$$

[Problem]

16. True

[Problem]

17. True

[Problem]

18.

$$\|A - B\| = 4$$

[Problem]

19. (a)

$$P = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}, \qquad \Sigma = \begin{bmatrix} 2\sqrt{2} & 0 \\ 0 & \sqrt{2} \end{bmatrix}, \qquad Q = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

(b) Plot to be added.

[Problem]

20.

$$\text{cond}(A) = \sqrt{\frac{5+\sqrt{17}}{5-\sqrt{17}}}$$

[Problem]

21. (a) $\text{cond}(A) = ||A|| \cdot ||A^{-1}|| = \frac{6}{2} = 3$

(b)

$$||A\mathbf{x}||_2 = 13, \qquad ||A\mathbf{x}||_1 = 19$$

(c)

$$B = \begin{bmatrix} 0 & \frac{1}{2} \\ 2 & 0 \end{bmatrix}.$$

[Problem]

22. *Under construction*

[Problem]

23. *Under construction*

[Problem]

24. *Under construction*

[Problem]

25. *Under construction*

[Problem]

26. *Under construction*

[Problem]

27. *Under construction*

[Problem]

28. *Under construction*

[Problem]

29. *Under construction*

[Problem]

30. *Under construction*

[Problem]

Principal Component Analysis

1. True

[Problem]

2. *Under construction*

[Problem]

3.

$$w_1 = \frac{1}{\sqrt{5}} \begin{bmatrix} 2 \\ -1 \end{bmatrix}, \qquad w_2 = \frac{1}{\sqrt{5}} \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

[Problem]

4.

$$\boldsymbol{w_1} = \frac{1}{\sqrt{10-2\sqrt{5}}}\begin{bmatrix} 2 \\ 1-\sqrt{5} \end{bmatrix}$$

[Problem]

5. False

[Problem]

6. False

[Problem]

7. False

[Problem]

8. True

[Problem]

9. False

[Problem]

10. False

[Problem]

Power Method

1. $c = 2$

[Problem]

2.

$$\lambda \approx 2.4, \qquad \boldsymbol{x} = \begin{bmatrix} 12/17 \\ 1 \\ 12/17 \end{bmatrix}$$

[Problem]

Discrete Fourier Transform

Complex Numbers, Vectors and Matrices

1. True

[Problem]

2. True

[Problem]

3. *Under construction*

[Problem]

4. *Under construction*

[Problem]

5. *Under construction*

[Problem]

Discrete Fourier Transform

1. True

[Problem]

2. True

[Problem]

3. False

[Problem]

4. True

[Problem]

5.
$$\langle \boldsymbol{x_1}, \boldsymbol{x_2} \rangle = -\frac{3}{8}$$

[Problem]

Sinusoids

1.
$$\mathrm{DFT}(\boldsymbol{x}) = \begin{bmatrix} 0 & 0 & 20i & 0 & 0 & 0 & -20i & 0 \end{bmatrix}^T$$

[Problem]

2.
$$A_0 = \frac{1}{8}, \qquad A_1 = \frac{1}{\sqrt{2}}, \qquad A_2 = \frac{1}{2}$$

and
$$k_1 = 2, \qquad k_2 = 3, \qquad \phi_1 = -\frac{\pi}{4}, \qquad \phi_2 = \frac{\pi}{3}$$

[Problem]

3. *Under construction*

[Problem]

4.
$$1 \to \mathrm{B}, \qquad 2 \to \mathrm{C}, \qquad 3 \to \mathrm{D}, \qquad 4 \to \mathrm{A}$$

[Problem]

5.
$$\boldsymbol{y} = 8\boldsymbol{e_0} - 4\boldsymbol{e_4} - 4\boldsymbol{e_{12}}$$

[Problem]

6. *Under construction*

[Problem]

7. False

[Problem]

8.
$$\boldsymbol{x} = \frac{1}{2} + 3cos(2\pi(1)\boldsymbol{t}) + 2cos(2\pi(5)\boldsymbol{t})$$

[Problem]

9. *Under construction*

[Problem]